Estonia Team Selection Test 2001

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by parmenides51

– Day 1

1 Consider on the coordinate plane all rectangles whose
(i) vertices have integer coordinates;
(ii) edges are parallel to coordinate axes;
(iii) area is 2^k, where $k = 0, 1, 2....$
Is it possible to color all points with integer coordinates in two colors so that no such rectangle has all its vertices of the same color?

2 Point X is taken inside a regular n-gon of side length a. Let $h_1, h_2, ..., h_n$ be the distances from X to the lines defined by the sides of the n-gon. Prove that $\frac{1}{h_1} + \frac{1}{h_2} + ... + \frac{1}{h_n} > \frac{2\pi}{a}$

3 Let k be a fixed real number. Find all functions $f : R \to R$ such that $f(x) + (f(y))^2 = kf(x + y^2)$ for all real numbers x and y.

– Day 2

4 Consider all products by $2, 4, 6, ..., 2000$ of the elements of the set $A = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{2000}, \frac{1}{2001}\right\}$. Find the sum of all these products.

5 Find the exponent of 37 in the representation of the number $111......11$ with $3 \cdot 37^{2000}$ digits equals to 1, as product of prime powers

6 Let C_1 and C_2 be the incircle and the circumcircle of the triangle ABC, respectively. Prove that, for any point A' on C_2, there exist points B' and C' such that C_1 and C_2 are the incircle and the circumcircle of triangle $A'B'C'$, respectively.