Estonia Team Selection Test 2002

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–	Day 1
1	The princess wishes to have a bracelet with r rubies and s emeralds arranged in such order that there exist two jewels on the bracelet such that starting with these and enumerating the jewels in the same direction she would obtain identical sequences of jewels. Prove that it is possible to fulfill the princesss wish if and only if r and s have a common divisor.
2	Consider an isosceles triangle KL_1L_2 with $\|KL_1\| = \|KL_2\|$ and let KA, L_1B_1, L_2B_2 be its angle bisectors. Prove that $\cos \angle B_1AB_2 < \frac{3}{5}$
3	In a certain country there are 10 cities connected by a network of one-way nonstop flights so that it is possible to fly (using one or more flights) from any city to any other. Let n be the least number of flights needed to complete a trip starting from one of the cities, visiting all others and returning to the starting point. Find the greatest possible value of n.
–	Day 2
4	Let $ABCD$ be a cyclic quadrilateral such that $\angle ACB = 2\angle CAD$ and $\angle ACD = 2\angle BAC$. Prove that $\|CA\| = \|CB\| + \|CD\|$.
5	Let $0 < a < \frac{\pi}{2}$ and $x_1, x_2, ..., x_n$ be real numbers such that $\sin x_1 + \sin x_2 + ... + \sin x_n \ge n \cdot sina$. Prove that $\sin(x_1 - a) + \sin(x_2 - a) + ... + \sin(x_n - a) \ge 0$.
6	Place a pebble at each *non-positive* integer point on the real line, and let n be a fixed positive integer. At each step we choose some n consecutive integer points, remove one of the pebbles located at these points and rearrange all others arbitrarily within these points (placing at most one pebble at each point). Determine whether there exists a positive integer n such that for any given $N > 0$ we can place a pebble at a point with coordinate greater than N in a finite number of steps described above.