Estonia Team Selection Test 2003

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by parmenides51

–	Day 1
1	Two treasure-hunters found a treasure containing coins of value $a_1 < a_2 < ... < a_{2003}$ (the quantity of coins of each value is unlimited). The first treasure-hunter forms all the possible sets of different coins containing odd number of elements, and takes the most valuable coin of each such set. The second treasure-hunter forms all the possible sets of different coins containing even number of elements, and takes the most valuable coin of each such set. Which one of them is going to have more money and how much more? (H. Nestra)
2	Let n be a positive integer. Prove that if the number overbrace $\underbrace{99...9}_{n}$ is divisible by n, then the number $\underbrace{11...1}_{n}$ is also divisible by n. (H. Nestra)
3	Let N be the set of all non-negative integers and for each $n \in N$ denote $n' = n+1$. The function $A : N^3 \to N$ is defined as follows: (i) $A(0, m, n) = m'$ for all $m, n \in N$ (ii) $A(k', 0, n) = \begin{cases} n & if\ k = 0 \\ 0 & if\ k = 1, \\ 1 & if\ k > 1 \end{cases}$ for all $k, n \in N$ (iii) $A(k', m', n) = A(k, A(k', m, n), n)$ for all $k, m, n \in N$. Compute $A(5, 3, 2)$. (H. Nestra)
–	Day 2
4	A deck consists of 2^n cards. The deck is shuffled using the following operation: if the cards are initially in the order $a_1, a_2, a_3, a_4, ..., a_{2^n-1}, a_{2^n}$ then after shuffling the order becomes $a_{2^{n-1}+1}, a_1, a_{2^{n-1}+2}, a_2$. Find the smallest number of such operations after which the original order of the cards is restored. (R. Palm)

5 Let a, b, c be positive real numbers satisfying the condition $\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}=1$. Prove the inequality

$$\frac{a}{\sqrt{1+a^2}}+\frac{b}{\sqrt{1+b^2}}+\frac{c}{\sqrt{1+c^2}}\le\frac{3\sqrt{3}}{2}$$

When does the equality hold?

(L. Parts)

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6 Let ABC be an acute-angled triangle, O its circumcenter and H its orthocenter. The orthogonal projection of the vertex A to the line BC lies on the perpendicular bisector of the segment AC. Compute $\frac{CH}{BO}$.

(J. Willemson)
