Estonia Team Selection Test 2004

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–	Day 1
1	Let $k > 1$ be a fixed natural number. Find all polynomials $P(x)$ satisfying the condition $P(x^k) = (P(x))^k$ for all real numbers x.
2	Let O be the circumcentre of the acute triangle ABC and let lines AO and BC intersect at point K. On sides AB and AC, points L and M are chosen such that $\|KL\| = \|KB\|$ and $\|KM\| = \|KC\|$. Prove that segments LM and BC are parallel.
3	For which natural number n is it possible to draw n line segments between vertices of a regular $2n$-gon so that every vertex is an endpoint for exactly one segment and these segments have pairwise different lengths?
–	Day 2
4	Denote $f(m) = \sum_{k=1}^m (-1)^k cos \frac{k\pi}{2m+1}$ For which positive integers m is $f(m)$ rational?
5	Find all natural numbers n for which the number of all positive divisors of the number lcm $(1, 2, ..., n)$ is equal to 2^k for some non-negative integer k.
6	Call a convex polyhedron a *footballoid* if it has the following properties. (1) Any face is either a regular pentagon or a regular hexagon. (2) All neighbours of a pentagonal face are hexagonal (a *neighbour* of a face is a face that has a common edge with it).