Estonia Team Selection Test 2005

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–	Day 1
1	On a plane, a line ℓ and two circles c_1 and c_2 of different radii are given such that ℓ touches both circles at point P. Point $M \neq P$ on ℓ is chosen so that the angle Q_1MQ_2 is as large as possible where Q_1 and Q_2 are the tangency points of the tangent lines drawn from M to c_i and c_2, respectively, differing from ℓ . Find $\angle PMQ_1 + \angle PMQ_2$
2	On the planet Automory, there are infinitely many inhabitants. Every Automorian loves exactly one Automorian and honours exactly one Automorian. Additionally, the following can be noticed: • each Automorian is loved by some Automorian; • if Automorian A loves Automorian B, then also all Automorians honouring A love B, •if Automorian A honours Automorian B, then also all Automorians loving A honour B. Is it correct to claim that every Automorian honours and loves the same Automorian?
3	Find all pairs (x, y) of positive integers satisfying the equation $(x + y)^x = x^y$.
–	Day 2
4	Find all pairs (a, b) of real numbers such that the roots of polynomials $6x^2 - 24x - 4a$ and $x^3 + ax^2 + bx - 8$ are all non-negative real numbers.
5	On a horizontal line, 2005 points are marked, each of which is either white or black. For every point, one finds the sum of the number of white points on the right of it and the number of black points on the left of it. Among the 2005 sums, exactly one number occurs an odd number of times. Find all possible values of this number.
6	Let Γ be a circle and let d be a line such that Γ and d have no common points. Further, let AB be a diameter of the circle Γ; assume that this diameter AB is perpendicular to the line d, and the point B is nearer to the line d than the point A. Let C be an arbitrary point on the circle Γ, different from the points A and B. Let D be the point of intersection of the lines AC and d. One of the two tangents from the point D to the circle Γ touches this circle Γ at a point E; hereby, we assume that the points B and E lie in the same halfplane with respect to the line AC. Denote by F the point of intersection of the lines BE and d. Let the line AF intersect the circle Γ at a point G, different from A. Prove that the reflection of the point G in the line AB lies on the line CF.