Estonia Team Selection Test 2008

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by parmenides51, orl

– Day 1

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1 There are 2008 participants in a programming competition. In every round, all programmers are divided into two equal-sized teams. Find the minimal number of rounds after which there can be a situation in which every two programmers have been in different teams at least once.

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2 Let $ABCD$ be a cyclic quadrangle whose midpoints of diagonals AC and BD are F and G, respectively.
a) Prove the following implication: if the bisectors of angles at B and D of the quadrangle intersect at diagonal AC then $\frac{1}{4}\cdot|AC|\cdot|BD| = |AG|\cdot|BF|\cdot|CG|\cdot|DF|$.
b) Does the converse implication also always hold?

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3 Let n be a positive integer, and let x and y be a positive real number such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\cdot\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)\cdot(1-y)}.$$

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– Day 2

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4 Sequence (G_n) is defined by $G_0 = 0, G_1 = 1$ and $G_n = G_{n-1} + G_{n-2} + 1$ for every $n \geq 2$. Prove that for every positive integer m there exist two consecutive terms in the sequence that are both divisible by m.

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5 Points A and B are fixed on a circle c_1. Circle c_2, whose centre lies on c_1, touches line AB at B. Another line through A intersects c_2 at points D and E, where D lies between A and E. Line BD intersects c_1 again at F. Prove that line EB is tangent to c_1 if and only if D is the midpoint of the segment BF.

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6 A *string of parentheses* is any word that can be composed by the following rules.
1) () is a string of parentheses.
2) If s is a string of parentheses then (s) is a string of parentheses.
3) If s and t are strings of parentheses then st is a string of parentheses.
The *midcode* of a string of parentheses is the tuple of natural numbers obtained by finding, for

all pairs of opening and its corresponding closing parenthesis, the number of characters remaining to the left from the medium position between these parentheses, and writing all these numbers in non-decreasing order. For example, the midcode of $(())$ is $(2, 2)$ and the midcode of ()() is $(1, 3)$. Prove that midcodes of arbitrary two different strings of parentheses are different.