

**HMMT Invitational Competition 2020**

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by a1267ab

- 1 Sir Alex is coaching a soccer team of  $n$  players of distinct heights. He wants to line them up so that for each player  $P$ , the total number of players that are either to the left of  $P$  and taller than  $P$  or to the right of  $P$  and shorter than  $P$  is even. In terms of  $n$ , how many possible orders are there?

*Michael Ren*

- 2 Some bishops and knights are placed on an infinite chessboard, where each square has side length 1 unit. Suppose that the following conditions hold:
- For each bishop, there exists a knight on the same diagonal as that bishop (there may be another piece between the bishop and the knight).
  - For each knight, there exists a bishop that is exactly  $\sqrt{5}$  units away from it.
  - If any piece is removed from the board, then at least one of the above conditions is no longer satisfied.

If  $n$  is the total number of pieces on the board, find all possible values of  $n$ .

*Sheldon Kieren Tan*

- 3 Let  $P_1P_2P_3P_4$  be a tetrahedron in  $\mathbb{R}^3$  and let  $O$  be a point equidistant from each of its vertices. Suppose there exists a point  $H$  such that for each  $i$ , the line  $P_iH$  is perpendicular to the plane through the other three vertices. Line  $P_1H$  intersects the plane through  $P_2, P_3, P_4$  at  $A$ , and contains a point  $B \neq P_1$  such that  $OP_1 = OB$ . Show that  $HB = 3HA$ .

*Michael Ren*

- 4 Let  $C_k = \frac{1}{k+1} \binom{2k}{k}$  denote the  $k^{\text{th}}$  Catalan number and  $p$  be an odd prime. Prove that exactly half of the numbers in the set

$$\left\{ \sum_{k=1}^{p-1} C_k n^k \mid n \in \{1, 2, \dots, p-1\} \right\}$$

are divisible by  $p$ .

*Tristan Shin*

- 5 A triangle and a circle are in the same plane. Show that the area of the intersection of the triangle and the circle is at most one third of the area of the triangle plus one half of the area of the circle.

*Krit Boonsiriseth*

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