AMC 10 2020

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1 What value of x satisfies

$$x - \frac{3}{4} = \frac{5}{12} - \frac{1}{3}?$$

(A) $-\frac{2}{3}$ **(B)** $\frac{7}{36}$ **(C)** $\frac{7}{12}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{5}{6}$

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2 The numbers $3, 5, 7, a,$ and b have an average (arithmetic mean) of 15. What is the average of a and b?

(A) 0 **(B)** 15 **(C)** 30 **(D)** 45 **(E)** 60

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3 Assuming $a \neq 3$, $b \neq 4$, and $c \neq 5$, what is the value in simplest form of the following expression?

$$\frac{a-3}{5-c} \cdot \frac{b-4}{3-a} \cdot \frac{c-5}{4-b}$$

(A) -1 **(B)** 1 **(C)** $\frac{abc}{60}$ **(D)** $\frac{1}{abc} - \frac{1}{60}$ **(E)** $\frac{1}{60} - \frac{1}{abc}$

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4 A driver travels for 2 hours at 60 miles per hour, during which her car gets 30 miles per gallon of gasoline. She is paid $\$0.50$ per mile, and her only expense is gasoline at $\$2.00$ per gallon. What is her net rate of pay, in dollars per hour, after this expense?

(A) 20 **(B)** 22 **(C)** 24 **(D)** 25 **(E)** 26

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5 What is the sum of all real numbers x for which $|x^2 - 12x + 34| = 2?$

(A) 12 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 25

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6 How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by $5?$

(A) 80 **(B)** 100 **(C)** 125 **(D)** 200 **(E)** 500

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7 The 25 integers from -10 to $14,$ inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?

(A) 2 **(B)** 5 **(C)** 10 **(D)** 25 **(E)** 50

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8 What is the value of

$$1+2+3-4+5+6+7-8+\cdots+197+198+199-200?$$

(A) $9,800$ **(B)** $9,900$ **(C)** $10,000$ **(D)** $10,100$ **(E)** $10,200$

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9 A single bench section at a school event can hold either 7 adults or 11 children. When N bench sections are connected end to end, an equal number of adults and children seated together will occupy all the bench space. What is the least possible positive integer value of N?

(A) 9 **(B)** 18 **(C)** 27 **(D)** 36 **(E)** 77

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10 Seven cubes, whose volumes are $1, 8, 27, 64, 125, 216$, and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?

(A) 644 **(B)** 658 **(C)** 664 **(D)** 720 **(E)** 749

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11 What is the median of the following list of 4040 numbers$?$

$$1, 2, 3, ..., 2020, 1^2, 2^2, 3^2, ..., 2020^2$$

(A) 1974.5 **(B)** 1975.5 **(C)** 1976.5 **(D)** 1977.5 **(E)** 1978.5

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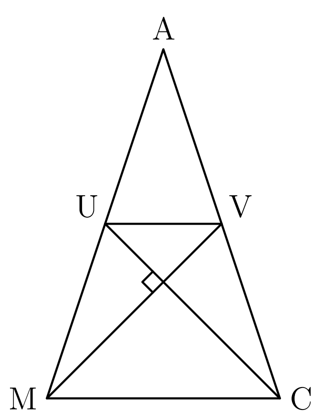
12 Triangle AMC is isoceles with $AM = AC$. Medians $\overline{MV}$ and $\overline{CU}$ are perpendicular to each other, and $MV = CU = 12$. What is the area of $\triangle AMC$?



(A) 48 **(B)** 72 **(C)** 96 **(D)** 144 **(E)** 192

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13 A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0, 0), (0, 4), (4, 4)$, and $(4, 0)$. What is the probability that the sequence of jumps ends on a vertical side of the square?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{8}$

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14 Real numbers x and y satisfy $x + y = 4$ and $x \cdot y = -2$. What is the value of

$$x + \frac{x^3}{y^2} + \frac{y^3}{x^2} + y?$$

(A) 360 **(B)** 400 **(C)** 420 **(D)** 440 **(E)** 480

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15 A positive integer divisor of $12!$ is chosen at random. The probability that the divisor chosen is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 3 **(B)** 5 **(C)** 12 **(D)** 18 **(E)** 23

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16 A point is chosen at random within the square in the coordinate plane whose vertices are $(0, 0), (2020, 0), (2020, 2020)$, and $(0, 2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x, y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

(A) 0.3 **(B)** 0.4 **(C)** 0.5 **(D)** 0.6 **(E)** 0.7

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17 Define

$$P(x) = (x - 1^2)(x - 2^2)\cdots(x - 100^2).$$

How many integers n are there such that $P(n) \leq 0$?

(A) 4900 **(B)** 4950 **(C)** 5000 **(D)** 5050 **(E)** 5100

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18 Let (a, b, c, d) be an ordered quadruple of not necessarily distinct integers, each one of them in the set $\{0, 1, 2, 3\}$. For how many such quadruples is it true that $a \cdot d - b \cdot c$ is odd? (For example, $(0, 3, 1, 1)$ is one such quadruple, because $0 \cdot 1 - 3 \cdot 1 = -3$ is odd.)

(A) 48 **(B)** 64 **(C)** 96 **(D)** 128 **(E)** 192

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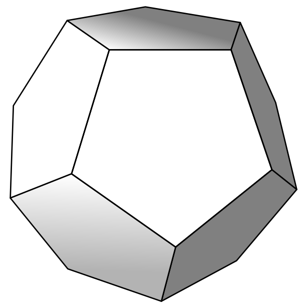
19 As shown in the figure below a regular dodecahedron (the polyhedron consisting of 12 congruent regular pentagonal faces) floats in space with two horizontal faces. Note that there is a ring of five slanted faces adjacent to the top face, and a ring of five slanted faces adjacent to the bottom face. How many ways are there to move from the top face to the bottom face via a sequence of adjacent faces so that each face is visited at most once and moves are not permitted from the bottom ring to the top ring?



(A) 125 **(B)** 250 **(C)** 405 **(D)** 640 **(E)** 810

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20 Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90^\circ$, $AC = 20$, and $CD = 30$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point E, and $AE = 5$. What is the area of quadrilateral $ABCD$?

(A) 330 **(B)** 340 **(C)** 350 **(D)** 360 **(E)** 370

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21 There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \; < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \; + 2^{a_k}.$$

What is k?

(A) 117 **(B)** 136 **(C)** 137 **(D)** 273 **(E)** 306

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22 For how many positive integers $n \le 1000$ is

$$\left\lfloor \frac{998}{n} \right\rfloor + \left\lfloor \frac{999}{n} \right\rfloor + \left\lfloor \frac{1000}{n} \right\rfloor$$

not divisible by 3? (Recall that $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

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23 Let T be the triangle in the coordinate plane with vertices $(0,0)$, $(4,0)$, and $(0,3)$. Consider the following five isometries (rigid transformations) of the plane: rotations of 90°, 180°, and 270° counterclockwise around the origin, reflection across the x-axis, and reflection across the y-axis. How many of the 125 sequences of three of these transformations (not necessarily distinct) will return T to its original position? (For example, a 180° rotation, followed by a reflection across the x-axis, followed by a reflection across the y-axis will return T to its original position, but a 90° rotation, followed by a reflection across the x-axis, followed by another reflection across the x-axis will not return T to its original position.)

(A) 12 **(B)** 15 **(C)** 17 **(D)** 20 **(E)** 25

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24 Let n be the least positive integer greater than 1000 for which

$$\gcd(63, n+120) = 21 \quad \text{and} \quad \gcd(n+63, 120) = 60.$$

What is the sum of the digits of n?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 24

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25 Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

(A) $\frac{7}{36}$ **(B)** $\frac{5}{24}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{17}{72}$ **(E)** $\frac{1}{4}$

– B

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1 What is the value of

$$1-(-2)-3-(-4)-5-(-6)?$$

(A) -20 **(B)** -3 **(C)** 3 **(D)** 5 **(E)** 21

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2 Carl has 5 cubes each having side length 1, and Kate has 5 cubes each having side length 2. What is the total volume of the 10 cubes?

(A) 24 **(B)** 25 **(C)** 28 **(D)** 40 **(E)** 45

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3 The ratio of w to x is $4:3$, the ratio of y to z is $3:2$, and the ratio of z to x is $1:6$. What is the ratio of w to y?

(A) $4:3$ **(B)** $3:2$ **(C)** $8:3$ **(D)** $4:1$ **(E)** $16:3$

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4 The acute angles of a right triangle are a° and b°, where $a > b$ and both a and b are prime numbers. What is the least possible value of b?

(A) 2 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 11

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5 How many distinguishable arrangements are there of 1 brown tile, 1 purple tile, 2 green tiles, and 3 yellow tiles in a row from left to right? (Tiles of the same color are indistinguishable)

(A) 210 **(B)** 420 **(C)** 630 **(D)** 840 **(E)** 1050

6 Driving along a highway, Megan noticed that her odometer showed 15951 (miles). This number is a palindromeit reads the same forward and backward. Then 2 hours later, the odometer displayed the next higher palindrome. What was her average speed, in miles per hour, during this 2-hour period?

(A) 50 **(B)** 55 **(C)** 60 **(D)** 65 **(E)** 70

7 How many positive even multiples of 3 less than 2020 are perfect squares?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 12

8 Points P and Q lie in a plane with $PQ = 8$. How many locations for point R in this plane are there such that the triangle with vertices P, Q, and R is a right triangle with area 12 square units?

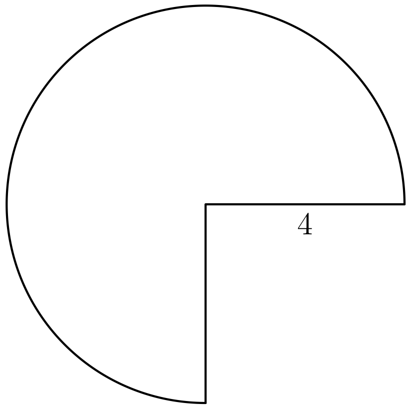
(A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 12

9 How many ordered pairs of integers (x, y) satisfy the equation

$$x^{2020} + y^2 = 2y?$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** infinitely many

10 A three-quarter sector of a circle of radius 4 inches together with its interior can be rolled up to form the lateral surface area of a right circular cone by taping together along the two radii shown. What is the volume of the cone in cubic inches?



(A) $3\pi\sqrt{5}$ **(B)** $4\pi\sqrt{3}$ **(C)** $3\pi\sqrt{7}$ **(D)** $6\pi\sqrt{3}$ **(E)** $6\pi\sqrt{7}$

11 Ms. Carr asks her students to read any 5 of the 10 books on a reading list. Harold randomly selects 5 books from this list, and Betty does the same. What is the probability that there are exactly 2 books that they both select?

(A) $\frac{1}{8}$ **(B)** $\frac{5}{36}$ **(C)** $\frac{14}{45}$ **(D)** $\frac{25}{63}$ **(E)** $\frac{1}{2}$

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12 The decimal representation of

$$\frac{1}{20^{20}}$$

consists of a string of zeros after the decimal point, followed by a 9 and then several more digits. How many zeros are in that initial string of zeros after the decimal point?

(A) 23 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 27

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13 Andy the Ant lives on a coordinate plane and is currently at $(-20, 20)$ facing east (that is, in the positive x-direction). Andy moves 1 unit and then turns 90° degrees left. From there, Andy moves 2 units (north) and then turns 90° degrees left. He then moves 3 units (west) and again turns 90° degrees left. Andy continues his progress, increasing his distance each time by 1 unit and always turning left. What is the location of the point at which Andy makes the 2020th left turn?

(A) $(-1030, -994)$ **(B)** $(-1030, -990)$ **(C)** $(-1026, -994)$ **(D)** $(-1026, -990)$ **(E)** $(-1022, -994$

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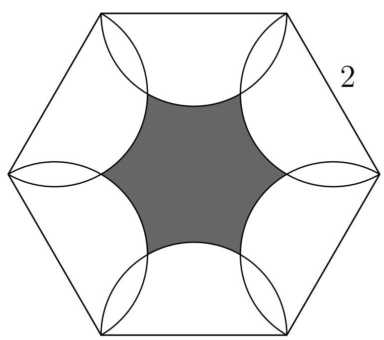
14 As shown in the figure below, six semicircles lie in the interior of a regular hexagon with side length 2 so that the diameters of the semicircles coincide with the sides of the hexagon. What is the area of the shaded regioninside the hexagon but outside all of the semicircles?



(A) $6\sqrt{3} - 3\pi$ **(B)** $\frac{9\sqrt{3}}{2} - 2\pi$ **(C)** $\frac{3\sqrt{3}}{2} - \frac{\pi}{3}$ **(D)** $3\sqrt{3} - \pi$
(E) $\frac{9\sqrt{3}}{2} - \pi$

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15 Steve wrote the digits 1, 2, 3, 4, and 5 in order repeatedly from left to right, forming a list of $10,000$ digits, beginning $123451234512\ldots$. He then erased every third digit from his list (that is, the 3rd, 6th, 9th, $\ldots$ digits from the left), then erased every fourth digit from the resulting list (that is, the 4th, 8th, 12th, $\ldots$ digits from the left in what remained), and then erased every fifth digit from what remained at that point. What is the sum of the three digits that were then in the positions $2019, 2020, 2021$?

(A) 7 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

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16 Bela and Jenn play the following game on the closed interval $[0, n]$ of the real number line, where n is a fixed integer greater than 4. They take turns playing, with Bela going first. At his first turn, Bela chooses any real number in the interval $[0, n]$. Thereafter, the player whose turn it is chooses a real number that is more than one unit away from all numbers previously chosen by either player. A player unable to choose such a number loses. Using optimal strategy, which player will win the game?

(A) Bela will always win. **(B)** Jenn will always win. **(C)** Bela will win if and only if n is odd. **(D)** Jenn will wir
(E) Jenn will win if and only if $n > 8$.

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17 There are 10 people standing equally spaced around a circle. Each person knows exactly 3 of the other 9 people: the 2 people standing next to her or him, as well as the person directly across the circle. How many ways are there for the 10 people to split up into 5 pairs so that the members of each pair know each other?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

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18 An urn contains one red ball and one blue ball. A box of extra red and blue balls lie nearby. George performs the following operation four times: he draws a ball from the urn at random and then takes a ball of the same color from the box and returns those two matching balls to the urn. After the four iterations the urn contains six balls. What is the probability that the urn contains three balls of each color?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

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19 In a certain card game, a player is dealt a hand of 10 cards from a deck of 52 distinct cards. The number of distinct (unordered) hands that can be dealt to the player can be written as $158A00A4AA0$. What is the digit A?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 7

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20 Let B be a right rectangular prism (box) with edges lengths $1, 3,$ and 4, together with its interior. For real $r \geq 0$, let $S(r)$ be the set of points in 3-dimensional space that lie within a distance r of some point B. The volume of $S(r)$ can be expressed as $ar^3 + br^2 + cr + d$, where a, b, c, and d are positive real numbers. What is $\frac{bc}{ad}$?

(A) 6 **(B)** 19 **(C)** 24 **(D)** 26 **(E)** 38

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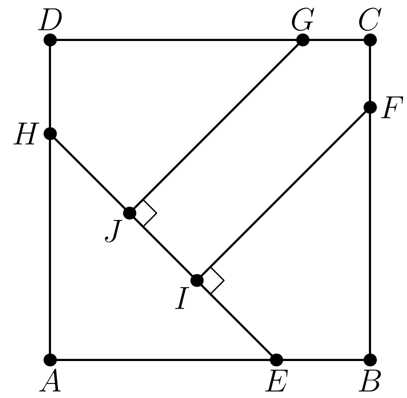
21 In square $ABCD$, points E and H lie on $\overline{AB}$ and $\overline{DA}$, respectively, so that $AE = AH$. Points F and G lie on $\overline{BC}$ and $\overline{CD}$, respectively, and points I and J lie on $\overline{EH}$ so that $\overline{FI} \perp \overline{EH}$ and $\overline{GJ} \perp \overline{EH}$. See the figure below. Triangle AEH, quadrilateral $BFIE$, quadrilateral $DHJG$, and pentagon $FCGJI$ each has area 1. What is FI^2?



(A) $\frac{7}{3}$ **(B)** $8-4\sqrt{2}$ **(C)** $1+\sqrt{2}$ **(D)** $\frac{7}{4}\sqrt{2}$ **(E)** $2\sqrt{2}$

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22 What is the remainder when $2^{202}+202$ is divided by $2^{101}+2^{51}+1$?

(A) 100 **(B)** 101 **(C)** 200 **(D)** 201 **(E)** 202

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23 Square $ABCD$ in the coordinate plane has vertices at the points $A(1,1), B(-1,1), C(-1,-1)$, and $D(1,-1)$. Consider the following four transformations:

-L, a rotation of 90° counterclockwise around the origin;
-R, a rotation of 90° clockwise around the origin;
-H, a reflection across the x-axis; and
-V, a reflection across the y-axis.

Each of these transformations maps the squares onto itself, but the positions of the labeled vertices will change. For example, applying R and then V would send the vertex A at $(1,1)$ to $(-1,-1)$ and would send the vertex B at $(-1,1)$ to itself. How many sequences of 20 transformations chosen from $\{L, R, H, V\}$ will send all of the labeled vertices back to their original positions? (For example, R, R, V, H is one sequence of 4 transformations that will send the vertices back to their original positions.)

(A) 2^{37} **(B)** $3\cdot 2^{36}$ **(C)** 2^{38} **(D)** $3\cdot 2^{37}$ **(E)** 2^{39}

24 How many positive integers n satisfy

$$\frac{n+1000}{70} = \lfloor \sqrt{n} \rfloor?$$

(Recall that $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

(A) 2 **(B)** 4 **(C)** 6 **(D)** 30 **(E)** 32

25 Let $D(n)$ denote the number of ways of writing the positive integer n as a product

$$n = f_1 \cdot f_2 \cdots f_k,$$

where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

(A) 112 **(B)** 128 **(C)** 144 **(D)** 172 **(E)** 184

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