Estonia Team Selection Test 2009

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by parmenides51

–	Day 1	

1 For arbitrary pairwise distinct positive real numbers a, b, c, prove the inequality

$$\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3} > 8abc$$

2 Call a finite set of positive integers *independent* if its elements are pairwise coprime, and *nice* if the arithmetic mean of the elements of every non-empty subset of it is an integer.
a) Prove that for any positive integer n there is an n-element set of positive integers which is both independent and nice.
b) Is there an infinite set of positive integers whose every independent subset is nice and which has an n-element independent subset for every positive integer n?

3 Find all natural numbers n for which there exists a convex polyhedron satisfying the following conditions:
(i) Each face is a regular polygon.
(ii) Among the faces, there are polygons with at most two different numbers of edges.
(iii) There are two faces with common edge that are both n-gons.

– Day 2

4 Points A', B', C' are chosen on the sides BC, CA, AB of triangle ABC, respectively, so that $\frac{|BA'|}{|A'C|} = \frac{|CB'|}{|B'A|} = \frac{|AC'|}{|C'B|}$. The line which is parallel to line $B'C'$ and goes through point A intersects the lines AC and AB at P and Q, respectively. Prove that $\frac{|PQ|}{|B'C'|} \ge 2$

5 A strip consists of n squares which are numerated in their order by integers $1, 2, 3, ..., n$. In the beginning, one square is empty while each remaining square contains one piece. Whenever a square contains a piece and its some neighbouring square contains another piece while the square immediately following the neighbouring square is empty, one may raise the first piece over the second one to the empty square, removing the second piece from the strip.
Find all possibilites which square can be initially empty, if it is possible to reach a state where the strip contains only one piece and
a) $n = 2008$,
b) $n = 2009$.

6 For any positive integer n, let $c(n)$ be the largest divisor of n not greater than $\sqrt{n}$ and let $s(n)$ be the least integer x such that $n < x$ and the product nx is divisible by an integer y where $n < y < x$. Prove that, for every n, $s(n) = (c(n) + 1) \cdot \left(\frac{n}{c(n)} + 1\right)$