Estonia Team Selection Test 2010

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by parmenides51

– Day 1

1 For arbitrary positive integers a, b, denote $a@b = \frac{a-b}{gcd(a,b)}$
Let n be a positive integer. Prove that the following conditions are equivalent:
(i) $gcd(n, n@m) = 1$ for every positive integer $m < n$,
(ii) $n = p^k$ where p is a prime number and k is a non-negative integer.

2 Let n be a positive integer. Find the largest integer N for which there exists a set of n weights such that it is possible to determine the mass of all bodies with masses of $1, 2, ..., N$ using a balance scale .
(i.e. to determine whether a body with unknown mass has a mass $1, 2, ..., N$, and which namely).

3 Let the angles of a triangle be α, β, and γ, the perimeter $2p$ and the radius of the circumcircle R. Prove the inequality $\cot^2\alpha + \cot^2\beta + \cot^2\gamma \ge 3\left(\frac{9R^2}{p^2} - 1\right)$. When is the equality achieved?

– Day 2

4 In an acute triangle ABC the angle C is greater than the angle A. Let AE be a diameter of the circumcircle of the triangle. Let the intersection point of the ray AC and the tangent of the circumcircle through the vertex B be K. The perpendicular to AE through K intersects the circumcircle of the triangle BCK for the second time at point D. Prove that CE bisects the angle BCD.

5 Let $P(x, y)$ be a non-constant homogeneous polynomial with real coefficients such that $P(\sin t, \cos t) = 1$ for every real number t. Prove that there exists a positive integer k such that $P(x, y) = (x^2 + y^2)^k$.

6 Every unit square of a $n \times n$ board is colored either red or blue so that among all 2 $\times 2$ squares on this board all possible colorings of 2×2 squares with these two colors are represented (colorings obtained from each other by rotation and reflection are considered different).
a) Find the least possible value of n.
b) For the least possible value of n find the least possible number of red unit squares