National Mathematical Olympiad 2004

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by parmenides51

–	2nd Round
1	Let m, n be integers so that $m \geq n > 1$. Let $F_1, ..., F_k$ be a collection of n-element subsets of $\{1, ..., m\}$ so that $F_i \cap F_j$ contains at most 1 element, $1 \leq i < j \leq k$. Show that $k \leq \frac{m(m-1)}{n(n-1)}$
2	Find the number of ordered pairs (a, b) of integers, where $1 \leq a, b \leq 2004$, such that $x^2+ax+b = 167y$ has integer solutions in x and y. Justify your answer.
3	Let AD be the common chord of two circles Γ_1 and Γ_2. A line through D intersects Γ_1 at B and Γ_2 at C. Let E be a point on the segment AD, different from A and D. The line CE intersect Γ_1 at P and Q. The line BE intersects Γ_2 at M and N. (i) Prove that P, Q, M, N lie on the circumference of a circle Γ_3. (ii) If the centre of Γ_3 is O, prove that OD is perpendicular to BC.
4	If $0 < x_1, x_2, ..., x_n \leq 1$, where $n \geq 1$, show that $$\frac{x_1}{1+(n-1)x_1} + \frac{x_2}{1+(n-1)x_2} + ... + \frac{x_n}{1+(n-1)x_n} \leq 1$$