Balkan MO Shortlist 2012

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– Algebra

A1 Prove that

$$\sum_{cyc}(x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx),$$

for all positive real numbers x, y and z.

A2 Let $a, b, c \geq 0$ and $a+b+c=\sqrt{2}$. Show that

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}} \geq 2+\frac{1}{\sqrt{3}}$$

In general if $a_1, a_2, \cdots, a_n \geq 0$ and $\sum_{i=1}^n a_i = \sqrt{2}$ we have

$$\sum_{i=1}^n \frac{1}{\sqrt{1+a_i^2}} \geq (n-1)+\frac{1}{\sqrt{3}}$$

A3 Determine the maximum possible number of distinct real roots of a polynomial $P(x)$ of degree 2012 with real coefficients satisfying the condition

$$P(a)^3+P(b)^3+P(c)^3 \geq 3P(a)P(b)P(c)$$

for all real numbers $a, b, c \in \mathbb{R}$ with $a+b+c=0$

A4 Let $ABCD$ be a square of the plane P. Define the minimum and the maximum the value of the function $f : P \to R$ is given by $f(P)=\frac{PA+PB}{PC+PD}$

A5 Let $f, g : \mathbb{Z} \to [0, \infty)$ be two functions such that $f(n)=g(n)=0$ with the exception of finitely many integers n. Define $h : \mathbb{Z} \to [0, \infty)$ by

$$h(n)=\max\{f(n-k)g(k) : k \in \mathbb{Z}\}.$$

Let p and q be two positive reals such that $1/p+1/q=1$. Prove that

$$\sum_{n\in\mathbb{Z}} h(n) \geq \left(\sum_{n\in\mathbb{Z}} f(n)^p\right)^{1/p}\left(\sum_{n\in\mathbb{Z}} g(n)^q\right)^{1/q}.$$

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A6 Let k be a positive integer. Find the maximum value of

$$a^{3k-1}b+b^{3k-1}c+c^{3k-1}a+k^2a^kb^kc^k,$$

where a, b, c are non-negative reals such that $a+b+c=3k$.

– Combinatorics

C1 Let n be a positive integer. Let $P_n=\{2^n,2^{n-1}\cdot 3,2^{n-2}\cdot 3^2,\dots,3^n\}$. For each subset X of P_n, we write S_X for the sum of all elements of X, with the convention that $S_\emptyset=0$ where $\emptyset$ is the empty set. Suppose that y is a real number with $0\le y\le 3^{n+1}-2^{n+1}$.
Prove that there is a subset Y of P_n such that $0\le y-S_Y<2^n$

– Geometry

G1 Let A, B and C be points lying on a circle Γ with centre O. Assume that $\angle ABC>90$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let l be the line through D which is perpendicular to AO. Let E be the point of intersection of l with the line AC, and let F be the point of intersection of Γ with l that lies between D and E.
Prove that the circumcircles of triangles BFE and CFD are tangent at F.

G2 Let ABC be a triangle, and let ℓ be the line passing through the circumcenter of ABC and parallel to the bisector of the angle $\angle A$. Prove that the line ℓ passes through the orthocenter of ABC if and only if $AB=AC$ or $\angle BAC=120^o$

G3 Let ABC be a triangle with circumcircle c and circumcenter O, and let D be a point on the side BC different from the vertices and the midpoint of BC. Let K be the point where the circumcircle c_1 of the triangle BOD intersects c for the second time and let Z be the point where c_1 meets the line AB. Let M be the point where the circumcircle c_2 of the triangle COD intersects c for the second time and let E be the point where c_2 meets the line AC. Finally let N be the point where the circumcircle c_3 of the triangle AEZ meets c again. Prove that the triangles ABC and NKM are congruent.

G4 Let M be the point of intersection of the diagonals of a cyclic quadrilateral $ABCD$. Let I_1 and I_2 are the incenters of triangles AMD and BMC, respectively, and let L be the point of intersection of the lines DI_1 and CI_2. The foot of the perpendicular from the midpoint T of I_1I_2 to CL is N, and F is the midpoint of TN. Let G and J be the points of intersection of the line LF with I_1N and I_1I_2, respectively. Let O_1 be the circumcenter of triangle LI_1J, and let Γ_1 and Γ_2 be the circles with diameters O_1L and O_1J, respectively. Let V and S be the second points of intersection of I_1O_1 with Γ_1 and Γ_2, respectively. If K is point where the circles Γ_1 and Γ_2 meet again, prove that K is the circumcenter of the triangle SVG.

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G5 $\boxed{\text{G5}}$ The incircle of a triangle ABC touches its sides BC,CA,AB at the points A_1,B_1,C_1.Let the projections of the orthocenter H_1 of the triangle $A_1B_1C_1$ to the lines AA_1 and BC be P and Q,respectively. Show that PQ bisects the line segment B_1C_1

G6 Let P and Q be points inside a triangle ABC such that $\angle PAC = \angle QAB$ and $\angle PBC = \angle QBA$. Let D and E be the feet of the perpendiculars from P to the lines BC and AC, and F be the foot of perpendicular from Q to the line AB. Let M be intersection of the lines DE and AB. Prove that $MP \perp CF$

G7 $ABCD$ is a cyclic quadrilateral. The lines AD and BC meet at X, and the lines AB and CD meet at Y . The line joining the midpoints M and N of the diagonals AC and BD, respectively, meets the internal bisector of angle AXB at P and the external bisector of angle BYC at Q. Prove that $PXQY$ is a rectangle

– Number Theory

N1 A sequence $(a_n)_{n=1}^{\infty}$ of positive integers satisfies the condition $a_{n+1} = a_n + \tau(n)$ for all positive integers n where $\tau(n)$ is the number of positive integer divisors of n. Determine whether two consecutive terms of this sequence can be perfect squares.

N2 Let the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ satisfy $a_0 = b_0 = 1, a_n = 9a_{n-1} - 2b_{n-1}$ and $b_n = 2a_{n-1} + 4b_{n-1}$ for all positive integers n. Let $c_n = a_n + b_n$ for all positive integers n.
Prove that there do not exist positive integers k, r, m such that $c_r^2 = c_k c_m$.

N3 Let $\mathbb{Z}^+$ be the set of positive integers. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the following conditions both hold:
(i) $f(n!) = f(n)!$ for every positive integer n,
(ii) $m - n$ divides $f(m) - f(n)$ whenever m and n are different positive integers.

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