2010 Balkan MO Shortlist

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– Algebra

A1 Let a, b and c be positive real numbers. Prove that

$$\frac{a^2b(b-c)}{a+b}+\frac{b^2c(c-a)}{b+c}+\frac{c^2a(a-b)}{c+a}\geq 0.$$

A2 Let the sequence $(a_n)_{n\in\mathbb{N}}$, where $\mathbb{N}$ denote the set of natural numbers, is given with $a_1=2$ and $a_{n+1}=a_n^2-a_n+1$. Find the minimum real number L, such that for every $k\in\mathbb{N}$

$$\sum_{i=1}^{k}\frac{1}{a_i}<L$$

A3 Let a, b, c, d be positive real numbers. Prove that

$$(\frac{a}{a+b})^5+(\frac{b}{b+c})^5+(\frac{c}{c+d})^5+(\frac{d}{d+a})^5\geq\frac{1}{8}$$

A4 Let $n>2$ be a positive integer. Consider all numbers S of the form

$$S=a_1a_2+a_2a_3+\ldots+a_{k-1}a_k$$

with $k>1$ and a_i begin positive integers such that $a_1+a_2+\ldots+a_k=n$. Determine all the numbers that can be represented in the given form.

– Combinatorics

C1 In a soccer tournament each team plays exactly one game with all others. The winner gets 3 points, the loser gets 0 and each team gets 1 point in case of a draw. It is known that n teams ($n\geq 3$) participated in the tournament and the final classification is given by the arithmetical progression of the points, the last team having only 1 point.

- Prove that this configuration is unattainable when $n=12$
- Find all values of n and all configurations when this is possible

C2 A grasshopper jumps on the plane from an integer point (point with integer coordinates) to another integer point according to the following rules: His first jump is of length $\sqrt{98}$, his second jump is of length $\sqrt{149}$, his next jump is of length $\sqrt{98}$, and so on, alternatively. What is the least possible odd number of moves in which the grasshopper could return to his starting point?

C3 A strip of width w is the set of all points which lie on, or between, two parallel lines distance w apart. Let S be a set of n ($n \geq 3$) points on the plane such that any three different points of S can be covered by a strip of width 1.
Prove that S can be covered by a strip of width 2.

C4 Integers are written in the cells of a table 2010×2010. Adding 1 to all the numbers in a row or in a column is called a *move*. We say that a table is *equilibrium* if one can obtain after finitely many moves a table in which all the numbers are equal.

-Find the largest positive integer n, for which there exists an *equilibrium* table containing the numbers $2^0, 2^1, \ldots, 2^n$.
- For this n, find the maximal number that may be contained in such a table.

C5 A train consist of 2010 wagons containing gold coins, all of the same shape. Any two coins have equal weight provided that they are in the same wagon, and differ in weight if they are in different ones. The weight of a coin is one of the positive reals

$$m_1 < m_2 < \ldots < m_{2010}$$

Each wagon is marked by a label with one of the numbers $m_1, m_2, \ldots, m_{2010}$ (the numbers on different labels are different).

A controller has a pair of scales (allowing only to compare masses) at his disposal. During each measurement he can use an arbitrary number of coins from any of the wagons. The controller has the task to establish: if all labels show rightly the common weight of the coins in a wagon or if there exists at least one wrong label. What is the least number of measurement that the controller has to perform to accomplish his task?

– Geometry

G1 Let $ABCDE$ be a pentagon with $\hat{A} = \hat{B} = \hat{C} = \hat{D} = 120^\circ$. Prove that $4 \cdot AC \cdot BD \geq 3 \cdot AE \cdot ED$.

G2 Consider a cyclic quadrilateral such that the midpoints of its sides form another cyclic quadrilateral. Prove that the area of the smaller circle is less than or equal to half the area of the bigger circle

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G3 The incircle of a triangle $A_0B_0C_0$ touches the sides B_0C_0, C_0A_0, A_0B_0 at the points A, B, C respectively, and the incircle of the triangle ABC with incenter I touches the sides BC, CA, AB

at the points A_1, B_1, C_1, respectively. Let $\sigma(ABC)$ and $\sigma(A_1B_1C)$ be the areas of the triangles ABC and A_1B_1C respectively. Show that if $\sigma(ABC) = 2\sigma(A_1B_1C)$, then the lines AA_0, BB_0, IC_1 pass through a common point .

G4 Let ABC be a given triangle and ℓ be a line that meets the lines BC, CA and AB in A_1, B_1 and C_1 respectively. Let A' be the midpoint, of the segment connecting the projections of A_1 onto the lines AB and AC. Construct, analogously the points B' and C'.
(a) Show that the points A', B' and C' are collinear on some line ℓ'.
(b) Show that if ℓ contains the circumcenter of the triangle ABC, then ℓ' contains the center of it's Euler circle.

G5 Let ABC be an acute triangle with orthocentre H, and let M be the midpoint of AC. The point C_1 on AB is such that CC_1 is an altitude of the triangle ABC. Let H_1 be the reflection of H in AB. The orthogonal projections of C_1 onto the lines AH_1, AC and BC are P, Q and R, respectively. Let M_1 be the point such that the circumcentre of triangle PQR is the midpoint of the segment MM_1.
Prove that M_1 lies on the segment BH_1.

G6 In a triangle ABC the excircle at the side BC touches BC in point D and the lines AB and AC in points E and F respectively. Let P be the projection of D on EF. Prove that the circumcircle k of the triangle ABC passes through P if and only if k passes through the midpoint M of the segment EF.

G7 A triangle ABC is given. Let M be the midpoint of the side AC of the triangle and Z the image of point B along the line BM. The circle with center M and radius MB intersects the lines BA and BC at the points E and G respectively. Let H be the point of intersection of EG with the line AC, and K the point of intersection of HZ with the line EB. The perpendicular from point K to the line BH intersects the lines BZ and BH at the points L and N, respectively.
If P is the second point of intersection of the circumscribed circles of the triangles KZL and BLN, prove that, the lines BZ, KN and HP intersect at a common point.

G8 Let $c(0, R)$ be a circle with diameter AB and C a point, on it different than A and B such that $\angle AOC > 90^o$. On the radius OC we consider the point K and the circle (c_1) with center K and radius $KC = R_1$. We draw the tangents AD and AE from A to the circle (c_1). Prove that the straight lines AC, BK and DE are concurrent

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– Number Theory

N1 Determine whether it is possible to partition $\mathbb{Z}$ into triples (a, b, c) such that, for every triple, $|a^3b + b^3c + c^3a|$ is perfect square.

N2 Solve the following equation in positive integers: $x^3 = 2y^2 + 1$

N3 For each integer n ($n \geq 2$), let $f(n)$ denote the sum of all positive integers that are at most n and not relatively prime to n.
Prove that $f(n+p) \neq f(n)$ for each such n and every prime p.