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Art of Problem Solving

2008 Balkan MO Shortlist

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– Algebra

A1 For all $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}^+$, Prove

$$\sum \frac{1}{2\nu\alpha_1+\alpha_2+\alpha_3} > \frac{2\nu}{2\nu+1}\left(\sum \frac{1}{\nu\alpha_1+\nu\alpha_2+\alpha_3}\right)$$

for every positive real number ν

A2 Is there a sequence $a_1, a_2, \dots$ of positive reals satisfying simoultaneously the following inequalities for all positive integers n:
a) $a_1+a_2+\ldots+a_n \le n^2$
b) $\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n} \le 2008$?

A3 Let (a_m) be a sequence satisfying $a_n \geq 0, n=0,1,2,\dots$ Suppose there exists $A>0$, $a_m - a_{m+1} \geq Aa_m^2$ for all $m \geq 0$. Prove that there exists $B>0$ such that

$$a_n \le \frac{B}{n} \qquad \text{for } 1 \le n$$

A4 We consider the set

$$\mathbb{C}^\nu = \{(z_1, z_2, \dots, z_\nu) \in \mathbb{C}\}, \qquad \nu \geq 2$$

and the function $\phi : \mathbb{C}^\nu \longrightarrow \mathbb{C}^\nu$ mapping every element $(z_1, z_2, \dots, z_\nu) \in \mathbb{C}^\nu$ to

$$\phi(z_1, z_2, \dots, z_\nu) = (z_1 - z_2, z_2 - z_3, \dots, z_\nu - z_1)$$

We also consider the $\nu-$tuple $(\omega_0, \omega_1, \dots, \omega_{\nu-1}) \in \mathbb{C}^\nu$ of the $n-$th roots of -1, where

$$\omega_\mu = \cos\left(\frac{\pi+2\mu\pi}{\nu}\right) + \iota \sin\left(\frac{\pi+2\mu\pi}{\nu}\right) \qquad \mu = 0, 1, \dots, \nu-1$$

Let after κ (where $\kappa \in \mathbb{N}$), successive applications of ϕ to the element $(\omega_0, \omega_1, \dots, \omega_{\nu-1})$, we obtain the element

$$\phi^{(\kappa)}(\omega_0, \omega_1, \dots, \omega_{\nu-1}) = (Z_{\kappa 1}, Z_{\kappa 2}, \dots, Z_{\kappa\nu})$$

Determine

- the values of ν for which all coordinates of $\phi^{(\kappa)}(\omega_0, \omega_1, \ldots, \omega_{\nu-1})$ have measures less than or equal to 1
- for $\nu = 4$, the minimal value of $\kappa \in \mathbb{N}$, for which

$$| Z_{\kappa i} | \geq 2^{100} \qquad 1 \leq i \leq 4$$

A5 Consider an integer $n \geq 1$, $a_1, a_2, \ldots, a_n$ real numbers in $[-1, 1]$ satisfying

$$a_1 + a_2 + \ldots + a_n = 0$$

and a function $f : [-1, 1] \mapsto \mathbb{R}$ such

$$| f(x) - f(y) | \leq | x - y |$$

for every $x, y \in [-1, 1]$. Prove

$$\left| f(x) - \frac{f(a_1) + f(a_2) + \ldots + f(a_n)}{n} \right| \leq 1$$

for every $x \in [-1, 1]$. For a given sequence $a_1, a_2, \ldots, a_n$, Find f and x so hat the equality holds.

A6 Prove that if $x, y, z \in \mathbb{R}^+$ such that xy, yz, zx are sidelengths of a triangle and $k \in [-1, 1]$, then

$$\sum \frac{\sqrt{xy}}{\sqrt{xz + yz + kxy}} \geq 2\sqrt{1 - k}$$

Determine the equality condition too.

A7 Let $x, y, z, t \in \mathbb{R}_{\geq 0}$. Show

$$\sqrt{xy} + \sqrt{xz} + \sqrt{xt} + \sqrt{yz} + \sqrt{yt} + \sqrt{zt} \geq 3\sqrt[3]{xyz + xyt + xzt + yzt}$$

and determine the equality cases.

– Combinatorics

C1 All $n+3$ offices of University of Somewhere are numbered with numbers $0, 1, 2, \ldots, n+1, n+2$ for some $n \in \mathbb{N}$. One day, Professor D came up with a polynomial with real coefficients and power n. Then, on the door of every office he wrote the value of that polynomial evaluated in the number assigned to that office. On the ith office, for $i \in \{0, 1, \ldots, n+1\}$ he wrote 2^i and on the $(n+2)$th office he wrote $2^{n+2} - n - 3$.

- Prove that Professor D made a calculation error
- Assuming that Professor D made a calculation error, what is the smallest number of errors he made? Prove that in this case the errors are uniquely determined, find them and correct them.

C2 In one of the countries, there are $n \geq 5$ cities operated by two airline companies. Every two cities are operated in both directions by at most one of the companies. The government introduced a restriction that all round trips that a company can offer should have atleast six cities. Prove that there are no more than $\lfloor \frac{n^2}{3} \rfloor$ flights offered by these companies.

C3 Let n be a positive integer. Consider a rectangle $(90n+1)\times(90n+5)$ consisting of unit squares. Let S be the set of the vertices of these squares. Prove that the number of distinct lines passing through at least two points of S is divisible by 4.

C4 An array $n \times n$ is given, consisting of n^2 unit squares. A pawn is placed arbitrarily on a unit square. A *move* of the pawn means a jump from a square of the kth column to any square of the kth row. Show that there exists a sequence of n^2 moves of the pawn so that all the unit squares of the array are visited once and, in the end, the pawn returns to the original position.

– Geometry

G1 In acute angled triangle ABC we denote by a, b, c the side lengths, by m_a, m_b, m_c the median lengths and by r_bc, r_{ca}, r_{ab} the radii of the circles tangents to two sides and to circumscribed circle of the triangle, respectively. Prove that

$$\frac{m_a^2}{r_{bc}} + \frac{m_b^2}{r_{ab}} + \frac{m_c^2}{r_{ab}} \geq \frac{27\sqrt{3}}{8}\sqrt[3]{abc}$$

G2 Given a scalene acute triangle ABC with $AC > BC$ let F be the foot of the altitude from C. Let P be a point on AB, different from A so that $AF = PF$. Let H, O, M be the orthocenter, circumcenter and midpoint of $[AC]$. Let X be the intersection point of BC and HP. Let Y be the intersection point of OM and FX and let OF intersect AC at Z. Prove that F, M, Y, Z are concyclic.

G3 We draw two lines $(\ell_1), (\ell_2)$ through the orthocenter H of the triangle ABC such that each one is dividing the triangle into two figures of equal area and equal perimeters. Find the angles of the triangle.

G4 A triangle ABC is given with barycentre G and circumcentre O. The perpendicular bisectors of GA, GB meet at C_1,of GB, GC meet at A_1, and GC, GA meet at B_1. Prove that O is the barycenter of the triangle $A_1B_1C_1$.

G5 The circle k_a touches the extensions of sides AB and BC, as well as the circumscribed circle of the triangle ABC (from the outside). We denote the intersection of k_a with the circumscribed circle of the triangle ABC by A'. Analogously, we define points B' and C'. Prove that the lines AA', BB' and CC' intersect in one point.

G6 On triangle ABC the AM ($M \in BC$) is median and BB_1 and CC_1 ($B_1 \in AC, C_1 \in AB$) are altitudes. The stright line d is perpendicular to AM at the point A and intersect the lines BB_1 and CC_1 at the points E and F respectively. Let denoted with ω the circle passing through the points E, M and F and with ω_1 and with ω_2 the circles that are tangent to segment EF and with ω at the arc EF which is not contain the point M. If the points P and Q are intersections points for ω_1 and ω_2 then prove that the points P, Q and M are collinear.

G7 In the non-isosceles triangle ABC consider the points X on $[AB]$ and Y on $[AC]$ such that $[BX] = [CY]$, M and N are the midpoints of the segments $[BC]$, respectively $[XY]$, and the straight lines XY and BC meet in K. Prove that the circumcircle of triangle KMN contains a point, different from M , which is independent of the position of the points X and Y.

G8 Let P be a point in the interior of a triangle ABC and let d_a, d_b, d_c be its distances to BC, CA, AB respectively. Prove that max $(AP, BP, CP) \ge \sqrt{d_a^2 + d_b^2 + d_c^2}$

– Number Theory

N1 Prove that for every natural number a, there exists a natural number that has the number a (the sequence of digits that constitute a) at its beginning, and which decreases a times when a is moved from its beginning to it end (any number zeros that appear in the beginning of the number obtained in this way are to be removed).

Example

- $a = 4$, then $\underline{4}10256 = 4 \cdot 10256\underline{4}$
- $a = 46$, then $\underline{46}0100021743857360295716 = 46 \cdot 10002174385736029571\underline{646}$

N2 Let c be a positive integer. The sequence $a_1, a_2, \ldots$ is defined as follows $a_1 = c$, $a_{n+1} = a_n^2 + a_n + c^3$ for all positive integers n. Find all c so that there are integers $k \ge 1$ and $m \ge 2$ so that $a_k^2 + c^3$ is the mth power of some integer.

N3 The sequence $(\chi_n)_{n=1}^{\infty}$ is defined as follows

$$\chi_{n+1} = \chi_n + \chi_{\lceil \frac{n}{2} \rceil} , \chi_1 = 1$$

Prove that none of the terms of this sequence are divisible by 4

N4 Solve the given equation in primes

$$xyz = 1 + 2^{y^2+1}$$

N5 Let (a_n) be a sequence with $a_1 = 0$ and $a_{n+1} = 2 + a_n$ for odd n and $a_{n+1} = 2a_n$ for even n. Prove that for each prime $p > 3$, the number

$$b = \frac{2^{2p} - 1}{3} \mid a_n$$

for infinitely many values of n

N6 Let (x_n), $n = 1, 2, \ldots$ be a sequence defined by $x_1 = 2008$ and

$$x_1 + x_2 + \ldots + x_{n-1} = \left(n^2 - 1\right) x_n \qquad \forall n \geq 2$$

Let the sequence $a_n = x_n + \frac{1}{n} S_n$, $n = 1, 2, 3, \ldots$ where $S_n = x_1 + x_2 + \ldots + x_n$. Determine the values of n for which the terms of the sequence a_n are perfect squares of an integer.
