

Estonia Team Selection Test 2012

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by parmenides51, Amir Hossein

– Day 1

1 Prove that for any positive integer k there exist k pairwise distinct integers for which the sum of their squares equals the sum of their cubes.

2 For a given positive integer n one has to choose positive integers a_0, a_1, \dots so that the following conditions hold:

- (1) $a_i = a_{i+n}$ for any i ,
- (2) a_i is not divisible by n for any i ,
- (3) a_{i+a_i} is divisible by a_i for any i .

For which positive integers $n > 1$ is this possible only if the numbers a_0, a_1, \dots are all equal?

3 In a cyclic quadrilateral $ABCD$ we have $|AD| > |BC|$ and the vertices C and D lie on the shorter arc AB of the circumcircle. Rays AD and BC intersect at point K , diagonals AC and BD intersect at point P . Line KP intersects the side AB at point L . Prove that $\angle ALK$ is acute.

– Day 2

4 Let ABC be a triangle where $|AB| = |AC|$. Points P and Q are different from the vertices of the triangle and lie on the sides AB and AC , respectively. Prove that the circumcircle of the triangle APQ passes through the circumcenter of ABC if and only if $|AP| = |CQ|$.

5 Let x, y, z be positive real numbers whose sum is 2012. Find the maximum value of

$$\frac{(x^2 + y^2 + z^2)(x^3 + y^3 + z^3)}{(x^4 + y^4 + z^4)}$$

6 Let m be a positive integer, and consider a $m \times m$ checkerboard consisting of unit squares. At the centre of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in the opposite directions meet, they both turn 90° clockwise and continue moving with speed 1. When more than 2 ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard, or prove that such a moment does not necessarily exist.

Proposed by Toomas Krips, Estonia
