Estonia Team Selection Test 2013

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–	Day 1
1	Find all prime numbers p for which one can find a positive integer m and nonnegative integers $a_0, a_1, ..., a_m$ less than p such that $$\begin{cases} a_0 + a_1 p + ... + a_{m-1} p^{m-1} + a_m p^m = 2013 \\ a_0 + a_1 + ... + a_{m-1} + a_m = 11 \end{cases}$$
2	For which positive integers $n \geq 3$ is it possible to mark n points of a plane in such a way that, starting from one marked point and moving on each step to the marked point which is the second closest to the current point, one can walk through all the marked points and return to the initial one? For each point, the second closest marked point must be uniquely determined.
3	Let $x_1, ..., x_n$ be non-negative real numbers, not all of which are zeros. (i) Prove that $$1 \leq \frac{\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + ... + \frac{x_n}{n}\right)(x_1 + 2x_2 + 3x_3 + ... + nx_n)}{(x_1 + x_2 + x_3 + ... + x_n)^2} \leq \frac{(n+1)^2}{4n}$$ (ii) Show that, for each $n > 1$, both inequalities can hold as equalities.
–	Day 2
4	Let D be the point different from B on the hypotenuse AB of a right triangle ABC such that $\|CB\| = \|CD\|$. Let O be the circumcenter of triangle ACD. Rays OD and CB intersect at point P, and the line through point O perpendicular to side AB and ray CD intersect at point Q. Points A, C, P, Q are concyclic. Does this imply that $ACPQ$ is a square?
5	Call a tuple $(b_m, b_{m+1}, ..., b_n)$ of integers perfect if both following conditions are fulfilled: 1. There exists an integer $a > 1$ such that $b_k = a^k + 1$ for all $k = m, m+1, ..., n$ 2. For all $k = m, m+1, ..., n$, there exists a prime number q and a non-negative integer t such that $b_k = q^t$. Prove that if $n - m$ is large enough then there is no perfect tuples, and find all perfect tuples with the maximal number of components.
6	A class consists of 7 boys and 13 girls. During the first three months of the school year, each boy has communicated with each girl at least once. Prove that there exist two boys and two girls such that both boys communicated with both girls first time in the same month.

