Estonia Team Selection Test 2014

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– Day 1

1 In Wonderland, the government of each country consists of exactly a men and b women, where a and b are fixed natural numbers and $b > 1$. For improving of relationships between countries, all possible working groups consisting of exactly one government member from each country, at least n among whom are women, are formed (where n is a fixed non-negative integer). The same person may belong to many working groups. Find all possibilities how many countries can be in Wonderland, given that the number of all working groups is prime.

2 Let a, b and c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a^2}{b^3 + c^4 + 1} + \frac{b^2}{c^3 + a^4 + 1} + \frac{c^2}{a^3 + b^4 + 1} > \frac{1}{5}$$

3 Three line segments, all of length 1, form a connected figure in the plane. Any two different line segments can intersect only at their endpoints. Find the maximum area of the convex hull of the figure.

– Day 2

4 In an acute triangle the feet of altitudes drawn from vertices A and B are D and E, respectively. Let M be the midpoint of side AB. Line CM intersects the circumcircle of CDE again in point P and the circumcircle of CAB again in point Q. Prove that $|MP| = |MQ|$.

5 In Wonderland there are at least 5 towns. Some towns are connected directly by roads or railways. Every town is connected to at least one other town and for any four towns there exists some direct connection between at least three pairs of towns among those four. When entering the public transportation network of this land, the traveller must insert one gold coin into a machine, which lets him use a direct connection to go to the next town. But if the traveller continues travelling from some town with the same method of transportation that took him there, and he has paid a gold coin to get to this town, then going to the next town does not cost anything, but instead the traveller gains the coin he last used back. In other cases he must pay just like when starting travelling. Prove that it is possible to get from any town to any other town by using at most 2 gold coins.

6 Find all natural numbers n such that the equation $x^2 + y^2 + z^2 = nxyz$ has solutions in positive integers

