Estonia Team Selection Test 2006

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– Day 1

1 Let k be any fixed positive integer. Let's look at integer pairs (a, b), for which the quadratic equations $x^2 - 2ax + b = 0$ and $y^2 + 2ay + b = 0$ are real solutions (not necessarily different), which can be denoted by x_1, x_2 and y_1, y_2, respectively, in such an order that the equation $x_1y_1 - x_2y_2 = 4k$.
a) Find the largest possible value of the second component b of such a pair of numbers (a, b).
b) Find the sum of the other components of all such pairs of numbers.

2 The center of the circumcircle of the acute triangle ABC is O. The line AO intersects BC at D. On the sides AB and AC of the triangle, choose points E and F, respectively, so that the points A, E, D, F lie on the same circle. Let E' and F' projections of points E and F on side BC respectively. Prove that length of the segment $E'F'$ does not depend on the position of points E and F.

3 A grid measuring 10×11 is given. How many "crosses" covering five unit squares can be placed on the grid?
(pictured right) so that no two of them cover the same square?
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– Day 2

4 The side AC of an acute triangle ABC is the diameter of the circle c_1 and side BC is the diameter of the circle c_2. Let E be the foot of the altitude drawn from the vertex B of the triangle and F the foot of the altitude drawn from the vertex A. In addition, let L and N be the points of intersection of the line BE with the circle c_1 (the point L lies on the segment BE) and the points of intersection of K and M of line AF with circle c_2 (point K is in section AF). Prove that $KLMN$ is a cyclic quadrilateral.

5 Let $a_1, a_2, a_3, ...$ be a sequence of positive real numbers. Prove that for any positive integer n the inequality holds $\sum_{i=1}^n b_i^2 \le 4\sum_{i=1}^n a_i^2$ where b_i is the arithmetic mean of the numbers $a_1, a_2, ..., a_n$

6 Denote by $d(n)$ the number of divisors of the positive integer n. A positive integer n is called highly divisible if $d(n) > d(m)$ for all positive integers $m < n$.
Two highly divisible integers m and n with $m < n$ are called consecutive if there exists no

highly divisible integer s satisfying $m < s < n$.

(a) Show that there are only finitely many pairs of consecutive highly divisible integers of the form (a, b) with $a \mid b$.

(b) Show that for every prime number p there exist infinitely many positive highly divisible integers r such that pr is also highly divisible.