Problems from the 2018-2019 Winter SDPC. Middle School division does 1,2,3,4,5, High School division does 3,4,5,6,7

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by mira74

1 Let r_1, r_2, r_3 be the distinct real roots of $x^3 - 2019x^2 - 2020x + 2021 = 0$. Prove that $r_1^3 + r_2^3 + r_3^3$ is an integer multiple of 3.

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2 Call a number *precious* if it is the sum of two distinct powers of two. Find all precious numbers n such that n^2 is also precious.

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3 A Pokemon Go player starts at $(0,0)$ and carries a pedometer that records the number of steps taken. He then takes steps with length 1 unit in the north, south, east, or west direction, such that each move after the first is perpendicular to the move before it. Somehow, the player eventually returns to $(0,0)$, but he had visited no point (except $(0,0)$) twice. Let n be the number on the pedometer when the player returns to $(0,0)$. Of the numbers from 1 to 2019 inclusive, how many can be the value of n?

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4 Tom is chasing Jerry on the coordinate plane. Tom starts at (x,y) and Jerry starts at $(0,0)$. Jerry moves to the right at 1 unit per second. At each positive integer time t, if Tom is within 1 unit of Jerry, he hops to Jerrys location and catches him. Otherwise, Tom hops to the midpoint of his and Jerrys location.

[i]Example. If Tom starts at $(3,2)$, then at time $t=1$ Tom will be at $(2,1)$ and Jerry will be at $(1,0)$. At $t=2$ Tom will catch Jerry.[/i]

Assume that Tom catches Jerry at some integer time n.
(a) Show that $x \geq 0$.
(b) Find the maximum possible value of $\frac{y}{x+1}$.

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5 Prove that there exists a positive integer N such that for every polynomial $P(x)$ of degree 2019, there exist N linear polynomials $p_1, p_2, \ldots p_N$ such that $P(x) = p_1(x)^{2019} + p_2(x)^{2019} + \ldots + p_N(x)^{2019}$. (Assume all polynomials in this problem have real coefficients, and leading coefficients cannot be zero.)

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6 Let S be the set of positive perfect squares that are of the form $\overline{AA}$, i.e. the concatenation of two equal integers A. (Integers are not allowed to start with zero.)

(a) Prove that S is infinite.
(b) Does there exist a function $f: S \times S \to S$ such that if $a, b, c \in S$ and $a, b | c$, then $f(a,b) | c$? (If such a function f exists, we call f an LCM function)

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7 In triangle ABC, let D be on side BC. The line through D parallel to AB, AC meet AC, AB at E, F, respectively.

(a) Show that if D varies on line BC, the circumcircle of AEF passes through a fixed point T.
(b) Show that if D lies on line AT, then the circumcircle of AEF is tangent to the circumcircle of BTC.

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