Online Math Open Problems 2020

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–	Spring
1	Let ℓ be a line and let points A, B, C lie on ℓ so that $AB = 7$ and $BC = 5$. Let m be the line through A perpendicular to ℓ. Let P lie on m. Compute the smallest possible value of $PB+PC$. *Proposed by Ankan Bhattacharya and Brandon Wang*
2	Po writes down five consecutive integers and then erases one of them. The four remaining integers sum to 153. Compute the integer that Po erased. *Proposed by Ankan Bhattacharya*
3	Given that the answer to this problem can be expressed as $a\cdot b\cdot c$, where a, b, and c are pairwise relatively prime positive integers with $b = 10$, compute $1000a + 100b + 10c$. *Proposed by Ankit Bisain*
4	Let $ABCD$ be a square with side length 16 and center O. Let $\mathcal{S}$ be the semicircle with diameter AB that lies outside of $ABCD$, and let P be a point on $\mathcal{S}$ so that $OP = 12$. Compute the area of triangle CDP. *Proposed by Brandon Wang*
5	Compute the smallest positive integer n such that there do not exist integers x and y satisfying $n = x^3 + 3y^3$. *Proposed by Luke Robitaille*
6	Alexis has 2020 paintings, the ith one of which is a $1\times i$ rectangle for $i = 1, 2, \dots, 2020$. Compute the smallest integer n for which they can place all of the paintings onto an $n \times n$ mahogany table without overlapping or hanging off the table. *Proposed by Brandon Wang*
7	On a 5×5 grid we randomly place two *cars*, which each occupy a single cell and randomly face in one of the four cardinal directions. It is given that the two cars do not start in the same cell. In a *move*, one chooses a car and shifts it one cell forward. The probability that there exists a sequence of moves such that, afterward, both cars occupy the same cell is $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $100m + n$. *Proposed by Sean Li*

8 Let $a > b$ be positive integers. Compute the smallest possible integer value of $\frac{a!+1}{b!+1}$.

Proposed by Sean Li

9 A magician has a hat that contains a white rabbits and b black rabbits. The magician repeatedly draws pairs of rabbits chosen at random from the hat, without replacement. Call a pair of rabbits *checkered* if it consists of one white rabbit and one black rabbit. Given that the magician eventually draws out all the rabbits without ever drawing out an unpaired rabbit and that the expected value of the number of checkered pairs that the magician draws is 2020, compute the number of possible pairs (a, b).

Proposed by Ankit Bisain

10 Compute the number of functions $f\colon \{1, \dots, 15\} \to \{1, \dots, 15\}$ such that, for all $x \in \{1, \dots, 15\}$,

$$\frac{f(f(x)) - 2f(x) + x}{15}$$

is an integer.

Proposed by Ankan Bhattacharya

11 A mahogany bookshelf has four identical-looking books which are 200, 400, 600, and 800 pages long. Velma picks a random book off the shelf, flips to a random page to read, and puts the book back on the shelf. Later, Daphne also picks a random book off the shelf and flips to a random page to read. Given that Velma read page 122 of her book and Daphne read page 304 of her book, the probability that they chose the same book is $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Sean Li

12 Convex pentagon $ABCDE$ is inscribed in circle γ. Suppose that $AB = 14$, $BE = 10$, $BC = CD = DE$, and $[ABCDE] = 3[ACD]$. Then there are two possible values for the radius of γ. The sum of these two values is $\sqrt{n}$ for some positive integer n. Compute n.

Proposed by Luke Robitaille

13 For nonnegative integers p, q, r, let

$$f(p, q, r) = (p!)^p (q!)^q (r!)^r.$$

Compute the smallest positive integer n such that for any triples (a, b, c) and (x, y, z) of nonnegative integers satisfying $a + b + c = 2020$ and $x + y + z = n$, $f(x, y, z)$ is divisible by $f(a, b, c)$.

Proposed by Brandon Wang

14 Let S and T be non-empty, finite sets of positive integers. We say that $a \in \mathbb{N}$ is *good* for $b \in \mathbb{N}$ if $a \geq \frac{b}{2} + 7$. We say that an ordered pair $(a, b) \in S \times T$ is *satisfiable* if a and b are good for each other.

A subset R of S is said to be *unacceptable* if there are less than $|R|$ elements b of T with the property that there exists $a \in R$ such that (a, b) is satisfiable. If there are no unacceptable subsets of S, and S contains the elements 14, 20, 16, 32, 23, and 31, compute the smallest possible sum of elements of T given that $|T| \geq 20$.

Proposed by Tristan Shin

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15 Let ABC be a triangle with $AB = 20$ and $AC = 22$. Suppose its incircle touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F respectively, and P is the foot of the perpendicular from D to $\overline{EF}$. If $\angle BPC = 90^\circ$, then compute BC^2.

Proposed by Ankan Bhattacharya

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16 Compute the number of ordered pairs (m, n) of positive integers such that $(2^m - 1)(2^n - 1) \mid 2^{10!} - 1$.

Proposed by Luke Robitaille

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17 Compute the number of integers $1 \leq n \leq 1024$ such that the sequence $\lceil n \rceil$, $\lceil n/2 \rceil$, $\lceil n/4 \rceil$, $\lceil n/8 \rceil$, $\ldots$ does not contain any multiple of 5.

Proposed by Sean Li

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18 Vincent has a fair die with sides labeled 1 to 6. He first rolls the die and records it on a piece of paper. Then, every second thereafter, he re-rolls the die. If Vincent rolls a different value than his previous roll, he records the value and continues rolling. If Vincent rolls the same value, he stops, does *not* record his final roll, and computes the average of his previously recorded rolls. Given that Vincent first rolled a 1, let E be the expected value of his result. There exist rational numbers $r, s, t > 0$ such that $E = r - s \ln t$ and t is not a perfect power. If $r + s + t = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Sean Li

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19 Let ABC be a scalene triangle. The incircle is tangent to lines BC, AC, and AB at points D, E, and F, respectively, and the A-excircle is tangent to lines BC, AC, and AB at points D_1, E_1, and F_1, respectively. Suppose that lines AD, BE, and CF are concurrent at point G, and suppose that lines AD_1, BE_1, and CF_1 are concurrent at point G_1. Let line GG_1 intersect the internal bisector of angle BAC at point X. Suppose that $AX = 1$, $\cos \angle BAC = \sqrt{3} - 1$, and $BC = 8\sqrt[4]{3}$. Then $AB \cdot AC = \frac{j + k\sqrt{m}}{n}$ for positive integers j, k, m, and n such that $\gcd(j, k, n) = 1$ and m is not divisible by the square of any integer greater than 1. Compute $1000j + 100k + 10m + n$.

Proposed by Luke Robitaille and Brandon Wang

20 Reimu invented a new number base system that uses exactly five digits. The number 0 in the decimal system is represented as 00000, and whenever a number is incremented, Reimu finds the leftmost digit (of the five digits) that is equal to the "units" (rightmost) digit, increments this digit, and sets all the digits to its right to 0. (For example, an analogous system that uses three digits would begin with 000, 100, 110, 111, 200, 210, 211, 220, 221, 222, 300, $\ldots$.) Compute the decimal representation of the number that Reimu would write as 98765.

Proposed by Yannick Yao

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21 For positive integers $i = 2, 3, \ldots, 2020$, let

$$a_i = \frac{\sqrt{3i^2+2i-1}}{i^3-i}.$$

Let x_2, $\ldots$, x_{2020} be positive reals such that $x_2^4 + x_3^4 + \cdots + x_{2020}^4 = 1 - \frac{1}{1010\cdot 2020\cdot 2021}$. Let S be the maximum possible value of

$$\sum_{i=2}^{2020} a_i x_i (\sqrt{a_i} - 2^{-2.25} x_i)$$

and let m be the smallest positive integer such that S^m is rational. When S^m is written as a fraction in lowest terms, let its denominator be $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ for prime numbers $p_1 < \cdots < p_k$ and positive integers α_i. Compute $p_1\alpha_1 + p_2\alpha_2 + \cdots + p_k\alpha_k$.

Proposed by Edward Wan and Brandon Wang

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22 Let ABC be a scalene triangle with incenter I and symmedian point K. Furthermore, suppose that $BC = 1099$. Let P be a point in the plane of triangle ABC, and let D, E, F be the feet of the perpendiculars from P to lines BC, CA, AB, respectively. Let M and N be the midpoints of segments EF and BC, respectively. Suppose that the triples (M, A, N) and (K, I, D) are collinear, respectively, and that the area of triangle DEF is 2020 times the area of triangle ABC. Compute the largest possible value of $\lceil AB + AC \rceil$.

Proposed by Brandon Wang

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23 In the Bank of Shower, a bored customer lays n coins in a row. Then, each second, the customer performs "The Process." In The Process, all coins with exactly one neighboring coin heads-up before The Process are placed heads-up (in its initial location), and all other coins are placed tails-up. The customer stops once all coins are tails-up.

Define the function f as follows: If there exists some initial arrangement of the coins so that the customer never stops, then $f(n) = 0$. Otherwise, $f(n)$ is the average number of seconds until the customer stops over all initial configurations. It is given that whenever $n = 2^k - 1$ for some positive integer k, $f(n) > 0$.

Let N be the smallest positive integer so that

$$M = 2^N \cdot \left(f(2^2-1)+f(2^3-1)+f(2^4-1)+\cdots+f(2^{10}-1)\right)$$

is a positive integer. If $M = \overline{b_k b_{k-1}\cdots b_0}$ in base two, compute $N+b_0+b_1+\cdots+b_k$.

Proposed by Edward Wan and Brandon Wang

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24 Let A, B be opposite vertices of a unit square with circumcircle Γ. Let C be a variable point on Γ. If $C \notin \{A,B\}$, then let ω be the incircle of triangle ABC, and let I be the center of ω. Let C_1 be the point at which ω meets $\overline{AB}$, and let D be the reflection of C_1 over line CI. If $C \in \{A,B\}$, let $D=C$. As C varies on Γ, D traces out a curve $\mathfrak{C}$ enclosing a region of area $\mathcal{A}$. Compute $\lfloor 10^4\mathcal{A}\rfloor$.

Proposed by Brandon Wang

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25 Let $\mathcal{S}$ denote the set of positive integer sequences (with at least two terms) whose terms sum to 2019. For a sequence of positive integers $a_1,a_2,\ldots,a_k$, its *value* is defined to be

$$V(a_1,a_2,\ldots,a_k) = \frac{a_1^{a_2}a_2^{a_3}\cdots a_{k-1}^{a_k}}{a_1!a_2!\cdots a_k!}.$$

Then the sum of the values over all sequences in $\mathcal{S}$ is $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute the remainder when $m+n$ is divided by 1000.

Proposed by Sean Li

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26 Let ABC be a triangle with circumcircle ω and circumcenter O. Suppose that $AB=15$, $AC=14$, and P is a point in the interior of $\triangle ABC$ such that $AP=\frac{13}{2}$, $BP^2=\frac{409}{4}$, and P is closer to $\overline{AC}$ than to $\overline{AB}$. Let E, F be the points where $\overline{BP}$, $\overline{CP}$ intersect ω again, and let Q be the intersection of $\overline{EF}$ with the tangent to ω at A. Given that $AQOP$ is cyclic and that CP^2 is expressible in the form $\frac{a}{b}-c\sqrt{d}$ for positive integers a, b, c, d such that $\gcd(a,b)=1$ and d is not divisible by the square of any prime, compute $1000a+100b+10c+d$.

Proposed by Edward Wan

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27 The *equatorial algebra* is defined as the real numbers equipped with the three binary operations $\natural$, $\sharp$, $\flat$ such that for all $x,y\in\mathbb{R}$, we have

$$x\natural y = x+y,\quad x\sharp y=\max\{x,y\},\quad x\flat y=\min\{x,y\}.$$

An *equatorial expression* over three real variables x, y, z, along with the *complexity* of such expression, is defined recursively by the following:

- x, y, and z are equatorial expressions of complexity 0;
- when P and Q are equatorial expressions with complexity p and q respectively, all of $P\natural Q$, $P\sharp Q$, $P\flat Q$ are equatorial expressions with complexity $1+p+q$.

Compute the number of distinct functions $f : \mathbb{R}^3 \to \mathbb{R}$ that can be expressed as equatorial expressions of complexity at most 3.

Proposed by Yannick Yao

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28 Let A_0BC_0D be a convex quadrilateral inscribed in a circle ω. For all integers $i \geq 0$, let P_i be the intersection of lines A_iB and C_iD, let Q_i be the intersection of lines A_iD and BC_i, let M_i be the midpoint of segment P_iQ_i, and let lines M_iA_i and M_iC_i intersect ω again at A_{i+1} and C_{i+1}, respectively. The circumcircles of $\triangle A_3M_3C_3$ and $\triangle A_4M_4C_4$ intersect at two points U and V.

If $A_0B = 3$, $BC_0 = 4$, $C_0D = 6$, $DA_0 = 7$, then UV can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c such that $\gcd(a, c) = 1$ and b is squarefree. Compute $100a + 10b + c$.

Proposed by Eric Shen

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29 Let x_0, x_1, …, x_{1368} be complex numbers. For an integer m, let $d(m)$, $r(m)$ be the unique integers satisfying $0 \leq r(m) < 37$ and $m = 37d(m) + r(m)$. Define the 1369×1369 matrix $A = \{a_{i,j}\}_{0 \leq i,j \leq 1368}$ as follows:

$$a_{i,j} = \begin{cases} x_{37d(j)+d(i)} & r(i) = r(j),\ i \neq j \\ -x_{37r(i)+r(j)} & d(i) = d(j),\ i \neq j \\ x_{38d(i)} - x_{38r(i)} & i = j \\ 0 & \text{otherwise} \end{cases}.$$

We say A is r-*murine* if there exists a 1369×1369 matrix M such that r columns of $MA - I_{1369}$ are filled with zeroes, where I_{1369} is the identity 1369×1369 matrix. Let $\mathrm{rk}(A)$ be the maximum r such that A is r-murine. Let S be the set of possible values of $\mathrm{rk}(A)$ as $\{x_i\}$ varies. Compute the sum of the 15 smallest elements of S.

Proposed by Brandon Wang

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30 Let c be the smallest positive real number such that for all positive integers n and all positive real numbers x_1, …, x_n, the inequality

$$\sum_{k=0}^{n} \frac{(n^3 + k^3 - k^2 n)^{3/2}}{\sqrt{x_1^2 + \cdots + x_k^2 + x_{k+1} + \cdots + x_n}} \leq \sqrt{3}\left(\sum_{i=1}^{n} \frac{i^3(4n - 3i + 100)}{x_i}\right) + cn^5 + 100n^4$$

holds. Compute $\lfloor 2020c \rfloor$.

Proposed by Luke Robitaille

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1 A circle with radius r has area 505. Compute the area of a circle with diameter $2r$.

Proposed by Luke Robitaille & Yannick Yao

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2 For any positive integer x, let $f(x) = x^x$. Suppose that n is a positive integer such that there exists a positive integer m with $m \neq 1$ such that $f(f(f(m))) = m^{m^{n+2020}}$. Compute the smallest possible value of n.

Proposed by Luke Robitaille

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3 Compute the number of ways to write the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 in the cells of a 3 by 3 grid such that

- each cell has exactly one number,

- each number goes in exactly one cell,

- the numbers in each row are increasing from left to right,

- the numbers in each column are increasing from top to bottom, and

-the numbers in the diagonal from the upper-right corner cell to the lower-left corner cell are increasing from upper-right to lower-left.

Proposed by Ankit Bisain & Luke Robitaille

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4 An alien from the planet OMO Centauri writes the first ten prime numbers in arbitrary order as U, W, XW, ZZ, V, Y, ZV, ZW, ZY, and X. Each letter represents a nonzero digit. Each letter represents the same digit everywhere it appears, and different letters represent different digits. Also, the alien is using a base other than base ten. The alien writes another number as UZWX. Compute this number (expressed in base ten, with the usual, human digits).

Proposed by Luke Robitaille & Eric Shen

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5 Compute the number of ordered triples of integers (a, b, c) between 1 and 12, inclusive, such that, if

$$q = a + \frac{1}{b} - \frac{1}{b + \frac{1}{c}},$$

then q is a positive rational number and, when q is written in lowest terms, the numerator is divisible by 13.

Proposed by Ankit Bisain

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6 Let x, y, and z be nonnegative real numbers with $x + y + z = 120$. Compute the largest possible value of the median of the three numbers $2x + y, 2y + z$, and $2z + x$.

Proposed by Ankit Bisain

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7 On a 9×9 square lake composed of unit squares, there is a 2×4 rectangular iceberg also composed of unit squares (it could be in either orientation; that is, it could be 4×2 as well). The sides of the iceberg are parallel to the sides of the lake. Also, the iceberg is invisible. Lily is trying to sink the iceberg by firing missiles through the lake. Each missile fires through a row or column, destroying anything that lies in its row or column. In particular, if Lily hits the iceberg with any missile, she succeeds. Lily has bought n missiles and will fire all n of them at once. Let N be the smallest possible value of n such that Lily can guarantee that she hits the iceberg. Let M be the number of ways for Lily to fire N missiles and guarantee that she hits the iceberg. Compute $100M + N$.

Proposed by Brandon Wang

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8 Let λ be a real number. Suppose that if $ABCD$ is any convex cyclic quadrilateral such that $AC = 4$, $BD = 5$, and $\overline{AB} \perp \overline{CD}$, then the area of $ABCD$ is at least λ. Then the greatest possible value of λ is $\frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

Proposed by Eric Shen

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9 Hong and Song each have a shuffled deck of eight cards, four red and four black. Every turn, each player places down the two topmost cards of their decks. A player can thus play one of three pairs: two black cards, two red cards, or one of each color. The probability that Hong and Song play exactly the same pairs as each other for all four turns is $\frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

Proposed by Sean Li

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10 Let $w, x, y,$ and z be nonzero complex numbers, and let n be a positive integer. Suppose that the following conditions hold:

- $\frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3,$
- $wx + wy + wz + xy + xz + yz = 14,$
- $(w + x)^3 + (w + y)^3 + (w + z)^3 + (x + y)^3 + (x + z)^3 + (y + z)^3 = 2160,$ and
- $w + x + y + z + i\sqrt{n} \in \mathbb{R}.$

Compute n.

Proposed by Luke Robitaille

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11 Let ABC be a triangle such that $AB = 5$, $AC = 8$, and $\angle BAC = 60^\circ$. Let P be a point inside the triangle such that $\angle APB = \angle BPC = \angle CPA$. Lines BP and AC intersect at E, and lines CP and AB intersect at F. The circumcircles of triangles BPF and CPE intersect at points P and $Q \neq P$. Then $QE + QF = \frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

Proposed by Ankan Bhattacharya

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12 At a party, there are 100 cats. Each pair of cats flips a coin, and they shake paws if and only if the coin comes up heads. It is known that exactly 4900 pairs of cats shook paws. After the party, each cat is independently assigned a "happiness index" uniformly at random in the interval $[0, 1]$. We say a cat is *practical* if it has a happiness index that is strictly greater than the index of every cat with which it shook paws. The expected value of the number of practical cats is $\frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

Proposed by Brandon Wang

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13 Let a, b, c, x, y, and z be positive integers such that

$$\frac{a^2 - 2}{x} = \frac{b^2 - 37}{y} = \frac{c^2 - 41}{z} = a + b + c.$$

Let $S = a + b + c + x + y + z$. Compute the sum of all possible values of S.

Proposed by Luke Robitaille

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14 Let $BCB'C'$ be a rectangle, let M be the midpoint of $B'C'$, and let A be a point on the circumcircle of the rectangle. Let triangle ABC have orthocenter H, and let T be the foot of the perpendicular from H to line AM. Suppose that $AM = 2$, $[ABC] = 2020$, and $BC = 10$. Then $AT = \frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

Proposed by Ankit Bisain

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15 Let m and n be positive integers such that $\gcd(m, n) = 1$ and

$$\sum_{k=0}^{2020} (-1)^k \binom{2020}{k} \cos(2020 \cos^{-1}(\tfrac{k}{2020})) = \frac{m}{n}.$$

Suppose n is written as the product of a collection of (not necessarily distinct) prime numbers. Compute the sum of the members of this collection. (For example, if it were true that $n = 12 = 2 \times 2 \times 3$, then the answer would be $2 + 2 + 3 = 7$.)

Proposed by Ankit Bisain

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16 For a positive integer n, we will say that a sequence $a_1, a_2, \dots a_n$ where $a_i \in \{1, 2, \dots, n\}$ for all i is n*-highly divisible* if, for every positive integer d that divides n and every nonnegative integer k less than $\frac{n}{d}$ we have that

$$d \left| \sum_{i=kd+1}^{(k+1)d} a_i. \right.$$

Let $\chi(n)$ be the probability that a sequence $a_1, a_2, \dots, a_n$ where a_i is chosen randomly from $\{1, 2, \dots n\}$ independently for all i is n-highly divisible. Suppose that n is a positive integer such that there exists a positive integer m not divisible by 3 such that $3^{40}\chi(n) = \frac{1}{m}$. Compute the sum of all possible values of n.

Proposed by Jaedon Whyte

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17 Let ABC be a triangle with $AB = 11, BC = 12$, and $CA = 13$, let M and N be the midpoints of sides CA and AB, respectively, and let the incircle touch sides CA and AB at points X and Y, respectively. Suppose that R, S, and T are the midpoints of line segments MN, BX, and CY, respectively. Then $\sin \angle SRT = \frac{k\sqrt{m}}{n}$, where k, m, and n are positive integers such that $\gcd(k, n) = 1$ and m is not divisible by the square of any prime. Compute $100k + 10m + n$.

Proposed by Tristan Shin

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18 The people in an infinitely long line are numbered $1, 2, 3, \dots$. Then, each person says either “Karl” or “Lark” independently and at random. Let S be the set of all positive integers i such that people i, $i+1$, and $i+2$ all say “Karl,” and define $X = \sum_{i \in S} 2^{-i}$. Then the expected value of X^2 is $\frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

Proposed by Ankit Bisain

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19 Compute the smallest positive integer M such that there exists a positive integer n such that

- M is the sum of the squares of some n consecutive positive integers, and
- $2M$ is the sum of the squares of some $2n$ consecutive positive integers.

Proposed by Jaedon Whyte

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20 Given a string of at least one character in which each character is either A or B, Kathryn is allowed to make these moves:

- she can choose an appearance of A, erase it, and replace it with BB, or

- she can choose an appearance of B, erase it, and replace it with AA.

Kathryn starts with the string A. Let a_n be the number of strings of length n that Kathryn can reach using a sequence of zero or more moves. (For example, $a_1 = 1$, as the only string of length 1 that Kathryn can reach is A.) Then $\sum_{n=1}^{\infty} \frac{a_n}{5^n} = \frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

Proposed by Luke Robitaille

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21 Among all ellipses with center at the origin, exactly one such ellipse is tangent to the graph of the curve $x^3 - 6x^2y + 3xy^2 + y^3 + 9x^2 - 9xy + 9y^2 = 0$ at three distinct points. The area of this ellipse is $\frac{k\pi\sqrt{m}}{n}$, where $k, m,$ and n are positive integers such that $\gcd(k, n) = 1$ and m is not divisible by the square of any prime. Compute $100k + 10m + n$.

Proposed by Jaedon Whyte

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22 Three points $P_1, P_2,$ and P_3 and three lines $\ell_1, \ell_2,$ and ℓ_3 lie in the plane such that none of the three points lie on any of the three lines. For (not necessarily distinct) integers i and j between 1 and 3 inclusive, we call a line ℓ (i, j)-*good* if the reflection of P_i across ℓ lies on ℓ_j, and call it *excellent* if there are two distinct pairs (i_1, j_1) and (i_2, j_2) for which it is good. Suppose that exactly N excellent lines exist. Compute the largest possible value of N.

Proposed by Yannick Yao

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23 For a positive integer $k > 1$ with $\gcd(k, 2020) = 1,$ we say a positive integer N is [i]k-bad[/i] if there do not exist nonnegative integers x and y with $N = 2020x + ky$. Suppose k is a positive integer with $k > 1$ and $\gcd(k, 2020) = 1$ such that the following property holds: if m and n are positive integers with $m+n = 2019(k-1)$ and $m \geq n$ and m is k-bad, then n is k-bad. Compute the sum of all possible values of k.

Proposed by Jaedon Whyte

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24 In graph theory, a *triangle* is a set of three vertices, every two of which are connected by an edge. For an integer $n \geq 3$, if a graph on n vertices does not contain two triangles that do not share any vertices, let $f(n)$ be the maximum number of triangles it can contain. Compute $f(3) + f(4) + \cdots + f(100)$.

Proposed by Edward Wan

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25 Let n be a positive integer with exactly twelve positive divisors $1 = d_1 < \cdots < d_{12} = n$. We say n is *trite* if

$$5 + d_6(d_6 + d_4) = d_7 d_4.$$

Compute the sum of the two smallest trite positive integers.

Proposed by Brandon Wang

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26 The bivariate functions $f_0, f_1, f_2, f_3, \ldots$ are sequentially defined by the relations $f_0(x, y) = 0$ and $f_{n+1}(x, y) = \big|x + |y + f_n(x, y)|\big|$ for all integers $n \geq 0$. For independently and randomly selected values $x_0, y_0 \in [-2, 2]$, let p_n be the probability that $f_n(x_0, y_0) < 1$. Let $a, b, c,$ and d be positive integers such that the limit of the sequence $p_1, p_3, p_5, p_7, \ldots$ is $\frac{\pi^2+a}{b}$ and the limit of the sequence $p_0, p_2, p_4, p_6, p_8, \ldots$ is $\frac{\pi^2+c}{d}$. Compute $1000a + 100b + 10c + d$.

Proposed by Sean Li

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27 Let ABC be a scalene, non-right triangle. Let ω be the incircle and let γ be the nine-point circle (the circle through the feet of the altitudes) of $\triangle ABC$, with centers I and N, respectively. Suppose ω and γ are tangent at a point F. Let D be the foot of the perpendicular from A to line BC and let M be the midpoint of side $\overline{BC}$. The common tangent to ω and γ at F intersects lines AB and AC at points P and Q, respectively. Let lines DP and DQ intersect γ at points $P_1 \neq D$ and $Q_1 \neq D$, respectively. Suppose that point Z lies on line P_1Q_1 such that $\angle MFZ = 90^\circ$ and $MZ \perp DF$. Suppose that γ has radius 11 and ω has radius 5. Then $DI = \frac{k\sqrt{m}}{n}$, where $k, m,$ and n are positive integers such that $\gcd(k, n) = 1$ and m is not divisible by the square of any prime. Compute $100k + 10m + n$.

Proposed by Luke Robitaille

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28 Julia bakes a cake in the shape of a unit square. Each minute, Julia makes two cuts through the cake as follows:

- she picks a **square** piece $\mathcal{S}$ of the cake with no cuts through its interior; then
- she slices the entire cake along the two lines parallel to the sides of the cake passing through the center of $\mathcal{S}$.

She does not move any pieces of cake during this process. After eight minutes, she has a grid of $9^2 = 81$ pieces of cake. (The pieces can be various sizes.) Compute the number of distinct grids that she could have ended up with. Two grids are the same if they have the same set of cuts; in particular, two grids that differ by a rotation or reflection are distinct.

Proposed by Sean Li

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29 Let ABC be a scalene triangle. Let $I_0 = A$ and, for every positive integer t, let I_t be the incenter of triangle $I_{t-1}BC$. Suppose that the points $I_0, I_1, I_2, \ldots$ all lie on some hyperbola $\mathcal{H}$ whose asymptotes are lines ℓ_1 and ℓ_2. Let the line through A perpendicular to line BC intersect ℓ_1 and ℓ_2 at points P and Q respectively. Suppose that $AC^2 = \frac{12}{7}AB^2 + 1$. Then the smallest possible value of the area of quadrilateral $BPCQ$ is $\frac{j\sqrt{k}+l\sqrt{m}}{n}$ for positive integers j, k, l, m, and n such that $\gcd(j, l, n) = 1$, both k and m are squarefree, and $j > l$. Compute $10000j + 1000k + 100l + 10m + n$.

Proposed by Gopal Goel, Luke Robitaille, Ashwin Sah, & Eric Shen

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30 Suppose that F is a field with exactly 5^{14} elements. We say that a function $f : F \to F$ is *happy*, if, for all $x, y \in F$,

$$(f(x+y)+f(x))\,(f(x-y)+f(x)) = f(y^2) - f(x^2).$$

Compute the number of elements z of F such that there exist distinct happy functions h_1 and h_2 such that $h_1(z) = h_2(z)$.

Proposed by Luke Robitaille
