Czech And Slovak Mathematical Olympiad, Round III, Category A 1987

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by byk7

1 Given a trapezoid, divide it by a line into two quadrilaterals in such a way that both of them are cyclic with the same circumradius. Discuss conditions of solvability.

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2 Given a prime $p > 3$ and an odd integer $n > 0$, show that the equation

$$xyz = p^n(x + y + z)$$

has at least $3(n + 1)$ different solutions up to symmetry. (That is, if (x', y', z') is a solution and (x'', y'', z'') is a permutation of the previous, they are considered to be the same solution.)

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3 Let $f : (0, \infty) \to (0, \infty)$ be a function satisfying $f\big(xf(y)\big) + f\big(yf(x)\big) = 2xy$ for all $x, y > 0$. Show that $f(x) = x$ for all positive x.

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4 Given an integer $n \geq 3$ consider positive integers $x_1, \dots, x_n$ such that $x_1 < x_2 < \cdots < x_n < 2x_1$. If p is a prime and r is a positive integer such that p^r divides the product $x_1 \cdots x_n$, prove that

$$\frac{x_1 \cdots x_n}{p^r} > n!.$$

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5 Consider a table with three rows and eleven columns. There are zeroes prefilled in the cell of the first row and the first column and in the cell of the second row and the last column. Determine the least real number α such that the table can be filled with non-negative numbers and the following conditions hold simultaneously:
(1) the sum of numbers in every column is one,
(2) the sum of every two neighboring numbers in the first row is at most one,
(3) the sum of every two neighboring numbers in the second row is at most one,
(4) the sum of every two neighboring numbers in the third row is at most α.

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6 Let AA', BB', CC' be parallel lines not lying in the same plane. Denote U the intersection of the planes $A'BC, AB'C, ABC'$ and V the intersection of the planes $AB'C', A'BC', A'B'C$. Show that the line UV is parallel with AA'.
