Czech And Slovak Mathematical Olympiad, Round III, Category A 1957

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by byk7

1 Find all real numbers p such that the equation

$$\sqrt{x^2 - 5p^2} = px - 1$$

has a root $x = 3$. Then, solve the equation for the determined values of p.

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2 Consider a (right) square pyramid $ABCDV$ with the apex V and the base (square) $ABCD$. Denote $d = AB/2$ and φ the dihedral angle between planes VAD and ABC.
(1) Consider a line XY connecting the skew lines VA and BC, where X lies on line VA and Y lies on line BC. Describe a construction of line XY such that the segment XY is of the smallest possible length. Compute the length of segment XY in terms of d, φ.
(2) Compute the distance v between points V and X in terms of d, φ.

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3 Find all real numbers α such that both values $\cot(\alpha)$ and $\cot(2\alpha)$ are integers.

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4 Consider a non-zero convex angle $\angle POQ$ and its inner point M. Moreover, let $m > 0$ be given. Construct a trapezoid $ABCD$ satisfying the following conditions:
(1) vertices A, D lie on ray OP and vertices lie on ray OQ,
(2) diagonals AC and BD intersect in M,
(3) $AB = m$.
Prove that your construction is correct and discuss conditions of solvability.
