Germany Team Selection Test 2011

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– VAIMO 1

1 Two circles ω,Ω intersect in distinct points A,B a line through B intersects ω,Ω in C,D respectively such that B lies between C,D another line through B intersects ω,Ω in E,F respectively such that E lies between B,F and $FE=CD$. Furthermore CF intersects ω,Ω in P,Q respectively and M,N are midpoints of the arcs PB,QB. Prove that $CNMF$ is a cyclic quadrilateral.

2 Let n be a positive integer prove that

$$6\nmid\lfloor(\sqrt[3]{28}-3)^{-n}\rfloor.$$

3 We call a function $f:\mathbb{Q}^+\to\mathbb{Q}^+$ *good* if for all $x,y\in\mathbb{Q}^+$ we have:

$$f(x)+f(y)\geq 4f(x+y).$$

a) Prove that for all good functions $f:\mathbb{Q}^+\to\mathbb{Q}^+$ and $x,y,z\in\mathbb{Q}^+$

$$f(x)+f(y)+f(z)\geq 8f(x+y+z)$$

b) Does there exists a good functions $f:\mathbb{Q}^+\to\mathbb{Q}^+$ and $x,y,z\in\mathbb{Q}^+$ such that

$$f(x)+f(y)+f(z)<9f(x+y+z)?$$

– VAIMO 2

1 A sequence $x_1,x_2,\ldots$ is defined by $x_1=1$ and $x_{2k}=-x_k, x_{2k-1}=(-1)^{k+1}x_k$ for all $k\geq 1$. Prove that $\forall n\geq 1$ $x_1+x_2+\ldots+x_n\geq 0$.

Proposed by Gerhard Wginger, Austria

2 Let $ABCDE$ be a convex pentagon such that $BC\parallel AE$, $AB=BC+AE$, and $\angle ABC=\angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO=90^\circ$, prove that $2\angle BDA=\angle CDE$.

Proposed by Nazar Serdyuk, Ukraine

3 Vertices and Edges of a regular n-gon are numbered $1, 2, \dots, n$ clockwise such that edge i lies between vertices $i, i+1 \mod n$. Now non-negative integers $(e_1, e_2, \dots, e_n)$ are assigned to corresponding edges and non-negative integers $(k_1, k_2, \dots, k_n)$ are assigned to corresponding vertices such that: i) $(e_1, e_2, \dots, e_n)$ is a permutation of $(k_1, k_2, \dots, k_n)$. ii) $k_i = |e_{i+1} - e_i|$ indexes $\mod n$.

a) Prove that for all $n \geq 3$ such non-zero n-tuples exist.
b) Determine for each m the smallest positive integer n such that there is an n-tuples stisfying the above conditions and also $\{e_1, e_2, \dots, e_n\}$ contains all $0, 1, 2, \dots m$.