Germany Team Selection Test 2012

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– VAIMO 1

1 Find the least integer k such that for any 2011×2011 table filled with integers Kain chooses, Abel be able to change at most k cells to achieve a new table in which 4022 sums of rows and columns are pairwise different.

2 Let Γ be the circumcircle of isosceles triangle ABC with vertex C. An arbitrary point M is chosen on the segment BC and point N lies on the ray AM with M between A, N such that $AN = AC$. The circumcircle of CMN cuts Γ in P other than C and AB, CP intersect at Q. Prove that $\angle BMQ = \angle QMN$.

3 Let a, b, c be positive real numbers with $a^2 + b^2 + c^2 \geq 3$. Prove that:

$$\frac{(a+1)(b+2)}{(b+1)(b+5)} + \frac{(b+1)(c+2)}{(c+1)(c+5)} + \frac{(c+1)(a+2)}{(a+1)(a+5)} \geq \frac{3}{2}.$$

– VAIMO 2

1 Consider a polynomial $P(x) = \prod_{j=1}^{9}(x+d_j)$, where $d_1, d_2, \ldots d_9$ are nine distinct integers. Prove that there exists an integer N, such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

Proposed by Luxembourg

2 Let ABC be an acute triangle. Let ω be a circle whose centre L lies on the side BC. Suppose that ω is tangent to AB at B' and AC at C'. Suppose also that the circumcentre O of triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

Proposed by Hrmel Nestra, Estonia

3 Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

Proposed by Japan