Problems from the 2018-2019 Fall SDPC. Middle School division does 1,2,3,5,6,7, High School division does 2,3,4,6,7,8.

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–	Session 1
1	An isosceles triangle T has the following property: it is possible to draw a line through one of the three vertices of T that splits it into two smaller isosceles triangles R and S, neither of which are similar to T. Find all possible values of the vertex (apex) angle of T.
2	Find all pairs of positive integers (m,n) such that $2^m - n^2$ is the square of an integer.
3	Let R be an 20×18 grid of points such that adjacent points are 1 unit apart. A fly starts at a point and jumps in straight lines to other points in R in turn, such that each point in R is visited exactly once and no two jumps intersect at a point other than an endpoint of a jump, for a total of 359 jumps. Call a jump small if it is of length 1. What is the least number of small jumps? (The left configuration for a 4×4 grid has 9 small jumps and 15 total jumps, while the right configuration is invalid.)
4	Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $$f(f(x) - f(y)) + 2f(xy) = x^2 f(x) + f(y^2)$$ for all real numbers x, y.
–	Session 2
5	For a positive integer that doesnt end in 0, define its reverse to be the number formed by reversing its digits. For instance, the reverse of 102304 is 403201. In terms of $n \geq 1$, how many numbers when added to its reverse give $10^n - 1$, the number consisting of n nines?
6	Alice and Bob play a game. Alice writes an equation of the form $ax^2 + bx + c = 0$, choosing a, b, c to be real numbers (possibly zero). Bob can choose to add (or subtract) any real number to each of a, b, c, resulting in a new equation. Bob wins if the resulting equation is quadratic and has two distinct real roots; Alice wins otherwise. For which choices of a, b, c does Alice win, no matter what Bob does?
7	The incircle of $\triangle ABC$ touches BC, CA, AB at D, E, F, respectively. Point P is chosen on EF such that AP is parallel to BC, and AD intersects the incircle of $\triangle ABC$ again at G. Show that $\angle AGP = 90^\circ$.

8 Let $S(n) = 1\varphi(1) + 2\varphi(2) \ldots + n\varphi(n)$, where $\varphi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n. (For instance $\varphi(12) = 4$ and $\varphi(20) = 8$.) Prove that for all $n \geq 2018$, the following inequality holds:

$$0.17n^3 \leq S(n) \leq 0.23n^3$$
