Germany Team Selection Test 2013

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by matinyousefi, lyukhson

– VAIMO 1

1 n is an odd positive integer and x,y are two rational numbers satisfying

$$x^n + 2y = y^n + 2x.$$

Prove that $x = y$.

2 Given a $m \times n$ grid rectangle with $m, n \geq 4$ and a closed path P that is not self intersecting from inner points of the grid, let A be the number of points on P such that P does not turn in them and let B be the number of squares that P goes through two non-adjacent sides of them furthermore let C be the number of squares with no side in P. Prove that

$$A = B - C + m + n - 1.$$

3 Let ABC be an acute-angled triangle with circumcircle ω. Prove that there exists a point J such that for any point X inside ABC if AX, BX, CX intersect ω in A_1, B_1, C_1 and A_2, B_2, C_2 be reflections of A_1, B_1, C_1 in midpoints of BC, AC, AB respectively then A_2, B_2, C_2, J lie on a circle.

– VAIMO 2

1 Two concentric circles ω, Ω with radii $8, 13$ are given. AB is a diameter of Ω and the tangent from B to ω touches ω at D. What is the length of AD.

2 Call admissible a set A of integers that has the following property:
If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k.
Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

Proposed by Warut Suksompong, Thailand

3 Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?