Germany Team Selection Test 2018

www.artofproblemsolving.com/community/c1128795

by matinyousefi, MarkBcc168, fastlikearabbit, math90, Muradjl

– VAIMO 1

1 A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

Proposed by Jeck Lim, Singapore

2 A positive integer d and a permutation of positive integers $a_1, a_2, a_3, \dots$ is given such that for all indices $i \geq 10^{100}$, $|a_{i+1} - a_i| \leq 2d$ holds. Prove that there exists infinity many indices j such that $|a_j - j| < d$.

3 In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

– VAIMO 2

1 Let $a_1, a_2, \dots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x + a_1)(x + a_2) \cdots (x + a_n)$$

has no positive roots.

2 Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

3 Determine all integers $n \geq 2$ having the following property: for any integers $a_1, a_2, \dots, a_n$ whose sum is not divisible by n, there exists an index $1 \leq i \leq n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \dots, a_i + a_{i+1} + \dots + a_{i+n-1}$$

is divisible by n. Here, we let $a_i = a_{i-n}$ when $i > n$.

Proposed by Warut Suksompong, Thailand