Germany Team Selection Test 2019

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– VAIMO 1

1 Let $\mathbb{Q}^+$ denote the set of all positive rational numbers. Determine all functions $f:\mathbb{Q}^+\to\mathbb{Q}^+$ satisfying
$$f(x^2f(y)^2)=f(x^2)f(y)$$
for all $x,y\in\mathbb{Q}^+$

2 Let ABC be a triangle with $AB=AC$, and let M be the midpoint of BC. Let P be a point such that $PB<PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM=\angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

3 Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n.
Prove that Sisyphus cannot reach the aim in less than
$$\left\lceil\frac{n}{1}\right\rceil+\left\lceil\frac{n}{2}\right\rceil+\left\lceil\frac{n}{3}\right\rceil+\dots+\left\lceil\frac{n}{n}\right\rceil$$
turns. (As usual, $\lceil x\rceil$ stands for the least integer not smaller than x.)

– VAIMO 2

1 Determine all pairs (n,k) of distinct positive integers such that there exists a positive integer s for which the number of divisors of sn and of sk are equal.

2 Does there exist a subset M of positive integers such that for all positive rational numbers $r<1$ there exists exactly one finite subset of M like S such that sum of reciprocals of elements in S equals r.

3 A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

Proposed by Mongolia