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W. 1 Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\sqrt[3]{\left(\frac{1+a}{b+c}\right)^{\frac{1-a}{bc}}\left(\frac{1+b}{c+a}\right)^{\frac{1-b}{ca}}\left(\frac{1+c}{a+b}\right)^{\frac{1-c}{ab}}} \geq 64$$

W. 2 Find the area of the set $A=\{(x,y)\,|\,1\leq x\leq e,\ 0\leq y\leq f(x)\}$, where $f(x)=\begin{vmatrix} 1 & 1 & 1 \\ \ln x & 2\ln x & 3\ln x \\ (\ln x)^2 & 4(\ln x)^2 & 9(\ln x)^2 \\ (\ln x)^3 & 8(\ln x)^3 & 27(\ln x)^3 \end{vmatrix}$

W. 3 Let Φ and Ψ denote the Euler totient and Dedekinds totient respectively. Determine all n such that $\Phi(n)$ divides $n+\Psi(n)$.

W. 4 Let Φ denote the Euler totient function. Prove that for infinitely many k we have $\Phi(2^k+1)<2^{k-1}$ and that for infinitely many m one has $\Phi(2^m+1)>2^{m-1}$

W. 5 Let p_1, p_2 be two odd prime numbers and α, n be positive integers with $\alpha>1, n>1$. Prove that if the equation $\left(\frac{p_2-1}{2}\right)^{p_1}+\left(\frac{p_2+1}{2}\right)^{p_1}=\alpha^n$ does not have integer solutions for both $p_1=p_2$ and $p_1\neq p_2$.

W. 6 Prove that

$$p(n)=2+\left(p(1)+\cdots+p\left(\left[\frac{n}{2}\right]+\chi_1(n)\right)+\left(p'_2(n)+\cdots+p'_{\left[\frac{n}{2}\right]-1}(n)\right)\right)$$

for every $n\in\mathbb{N}$ with $n>2$ where χ denotes the principal character Dirichlet modulo 2, i.e.

$$\chi_1(n)=\begin{cases}1 & \text{if } (n,2)=1\\ 0 & \text{if } (n,2)>1\end{cases}$$

with $p(n)$ we denote number of possible partitions of n and $p'_m(n)$ we denote the number of partitions of n in exactly m sumands.

W. 7 If $0<a<b$ then

$$\int_a^b \frac{\left(x^2-\left(\frac{a+b}{2}\right)^2\right)\ln\frac{x}{a}\ln\frac{x}{b}}{(x^2+a^2)(x^2+b^2)}dx>0$$

W. 8 If $n, p, q \in \mathbb{N}, p < q$ then

$$\binom{(p+q)n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{(p+q-1)n}{pn-k}=\binom{(p+q)n}{pn}\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{pn}{k}\binom{(q-p)n}{n-2k}$$

W. 9 Let the series

$$s(n,x)=\sum_{k=0}^{n}\frac{(1-x)(1-2x)(1-3x)\cdots(1-nx)}{n!}$$

Find a real set on which this series is convergent, and then compute its sum. Find also

$$\lim_{(n,x)\to(\infty,0)} s(n,x)$$

W. 10 Let consider the following function set

$$F=\{f \mid f:\{1,\ 2,\ \cdots,\ n\}\to\{1,\ 2,\ \cdots,\ n\}\}$$

- Find $|F|$
- For $n=2k$ prove that $|F|<e(4k)^k$
- Find n, if $|F|=540$ and $n=2k$

W. 11 Find all real numbers m such that

$$\frac{1-m}{2m}\in\{x \mid m^2x^4+3mx^3+2x^2+x=1\ \forall\, x\in\mathbb{R}\}$$

W. 12 Find all functions $f:(0,+\infty)\cap\mathbb{Q}\to(0,+\infty)\cap\mathbb{Q}$ satisfying thefollowing conditions:

- $f(ax)\leq(f(x))^a$, for every $x\in(0,+\infty)\cap\mathbb{Q}$ and $a\in(0,1)\cap\mathbb{Q}$
- $f(x+y)\leq f(x)f(y)$, for every $x,y\in(0,+\infty)\cap\mathbb{Q}$

W. 13 If $a_k>0$ [k =1, 2, $\cdots$, n], then prove the following inequality

$$\left(\sum_{k=1}^{n}a_k^5\right)^4\geq\frac{1}{n}\left(\frac{2}{n-1}\right)^5\left(\sum_{1\leq i<j\leq n}a_i^2a_j^2\right)^5$$

W. 14 If the function $f:[0,1]\to(0.+\infty)$ is increasing and continuous, then for every $a\ge 0$ the following inequality holds:

$$\int_0^1 \frac{x^{a+1}}{f(x)}dx \le \frac{a+1}{a+2}\int_0^1 \frac{x^a}{f(x)}dx$$

W. 15 Let a triangle $\triangle ABC$ and the real numbers x, y, $z>0$. Prove that

$$x^n\cos\frac{A}{2}+y^n\cos\frac{B}{2}+z^n\cos\frac{C}{2}\ge (yz)^{\frac{n}{2}}\sin A+(zx)^{\frac{n}{2}}\sin B+(xy)^{\frac{n}{2}}\sin C$$

W. 16 Prove that

$$\sum_{k=1}^{n}\frac{1}{d(k)}>\sqrt{n+1}-1$$

For every $n\ge 1$, $d(n)$ is the number of divisors of n

W. 17 If a, b, $c>0$ and $abc=1$, $\alpha=max\{a,b,c\}$; $f,g:(0,+\infty)\to\mathbb{R}$, where $f(x)=\frac{2(x+1)^2}{x}$ and $g(x)=(x+1)\left(\frac{1}{\sqrt{x}}+1\right)^2$, then

$$(a+1)(b+1)(c+1)\ge min\{\{f(x),g(x)\}\mid x\in\{a,b,c\}\backslash\{\alpha\}\}$$

W. 18 If a, b, $c>0$ and $abc=1$, then

$$\sum^{cyc}\frac{a+b+c^n}{a^{2n+3}+b^{2n+3}+ab}\le a^{n+1}+b^{n+1}+c^{n+1}$$

for all $n\in\mathbb{N}$

W. 19 If $x_k>0$ ($k=1,2,\cdots,n$), then

$$\sum_{k=1}^{n}\left(\frac{x_k}{1+x_1^2+x_2^2+\cdots+x_k^2}\right)^2\le\frac{\sum\limits_{k=1}^{n}x_k^2}{1+\sum\limits_{k=1}^{n}x_k^2}$$

W. 20 If $x\in\mathbb{R}\backslash\left\{\frac{k\pi}{2}\mid k\in\mathbb{Z}\right\}$, then

$$\left(\sum_{0\le j<k\le n}\sin(2(j+k)x)\right)^2+\left(\sum_{0\le j<k\le n}\cos(2(j+k)x)\right)^2=\frac{\sin^2 nx\sin^2(n+1)x}{\sin^2 x\sin^2 2x}$$

W. 21 If ζ denote the Riemann Zeta Function, and $s>1$ then

$$\sum_{k=1}^{\infty} \frac{1}{1+k^s} \geq \frac{\zeta(s)}{1+\zeta(s)}$$

W. 22 If $a_i > 0$ $(i=1,2,\cdots,n)$, then

$$\left(\frac{a_1}{a_2}\right)^k + \left(\frac{a_2}{a_3}\right)^k + \cdots + \left(\frac{a_n}{a_1}\right)^k \geq \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1}$$

for all $k \in \mathbb{N}$

W. 23 If $x_k \in \mathbb{R}$ $(k=1,2,\cdots,n)$ and $m \in \mathbb{N}$ then

- $\sum\limits_{cyc} \left(x_1^2 - x_1x_2 + x_2^2\right)^m \leq 3^m \sum\limits_{k=1}^{n} x_k^{2m}$
- $\prod\limits_{cyc} \left(x_1^2 - x_1x_2 + x_2^2\right)^m \leq \left(\frac{3^m}{n}\right)^m \left(\sum\limits_{k=1}^{n} x_k^{2m}\right)^n$

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w. 24 If K, L, M denote the midpoints of the sides AB, BC, CA in triangle $\triangle ABC$, then for all P in the plane of triangle $\triangle ABC$, we have

$$\frac{AB}{PK} + \frac{BC}{PL} + \frac{CA}{PM} \geq \frac{AB \cdot BC \cdot CA}{4 \cdot PK \cdot PL \cdot PM}$$

W. 25 Let $ABCD$ be a quadrilateral in which $\widehat{A} = \widehat{C} = 90^\circ$. Prove that

$$\frac{1}{BD}(AB+BC+CD+DA) + BD^2\left(\frac{1}{AB \cdot AD} + \frac{1}{CB \cdot CD}\right) \geq 2\left(2+\sqrt{2}\right)$$

W. 26 If $a_i > 0$ $(i=1,2,\cdots,n)$ and $\sum\limits_{i=1}^{n} a_i^k = 1$, where $1 \leq k \leq n+1$, then

$$\sum_{i=1}^{n} a_i + \frac{1}{\prod\limits_{i=1}^{n} a_i} \geq n^{1-\frac{1}{k}} + n^{\frac{n}{k}}$$

W. 27 Let a, n be positive integers such that a^n is a perfect number. Prove that

$$a^{\frac{n}{\mu}} > \frac{\mu}{2}$$

where μ denotes the number of distinct prime divisors of a^n

W. 28 Let θ and $p(p < 1)$) be nonnegative real numbers.

Suppose that $f : X \to Y$ is mapping with $f(0) = 0$ and

$$\left|\left|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right|\right|_Y \leq \theta\left(||x||_X^p + ||y||_X^p\right)$$

for all x, $y \in \mathbb{Z}$ with $x \perp y$ where X is an orthogonality space and Y is a real Banach space.

Prove that there exists a unique orthogonally Jensen additive mapping $T : X \to Y$, namely a mapping T that satisfies the so-called orthogonally Jensen additive functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$$

for all x, $y \in \mathbb{X}$ with $x \perp y$, satisfying the property

$$||f(x) - T(x)||_Y \leq \frac{2^p\theta}{2-2^p}||x||_X^p$$

for all $x \in X$

W. 29 Prove that for all triangle $\triangle ABC$ holds the following inequality

$$\sum_{cyc}\left(1 - \sqrt{\sqrt{3}\tan\frac{A}{2}} + \sqrt{3}\tan\frac{A}{2}\right)\left(1 - \sqrt{\sqrt{3}\tan\frac{B}{2}} + \sqrt{3}\tan\frac{B}{2}\right) \geq 3$$

W. 30 Prove that

$$\sum_{0\leq i<j\leq n}(i+j)\binom{n}{i}\binom{n}{j} = n\left(2^{2n-1} - \binom{2n-1}{n}\right)$$
