Tournament Of Towns 1988

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by parmenides51

– Spring 1988

– Junior

– Training

(164) 1 In January Kolya and Vasya have been assessed at school 20 times and each has been given 20 marks (each being an integer no greater than 5 , with both Kolya and Vasya receiving at least twos on each occasion). Kolya has been given as many fives as Vasya fours, as many fours as Vasya threes, as many threes as Vasya twos and as many twos as Vasya fives. If each has the same average mark , determine how many twos were given to Kolya.

(S . Fomin, Leningrad)

(165) 2 We are given convex quadrilateral $ABCD$. The midpoints of BC and DA are M and N respectively. The diagonal AC divides MN in half. Prove that the areas of triangles ABC and ACD are equal .

(166) 3 (a) The vertices of a regular 10-gon are painted in turn black and white. Two people play the following game . Each in turn draws a diagonal connecting two vertices of the same colour . These diagonals must not intersect . The winner is the player who is able to make the last move. Who will win if both players adopt the best strategy?
(b) Answer the same question for the regular 12-gon .

(V.G. Ivanov)

(167) 4 The numbers from 1 to 64 are written on the squares of a chessboard (from 1 to 8 from left to right on the first row , from 9 to 16 from left to right on the second row , and so on). Pluses are written before some of the numbers, and minuses are written before the remaining numbers in such a way that there are 4 pluses and 4 minuses in each row and in each column . Prove that the sum of the written numbers is equal to zero.

– Main

(168) 1 We are given that a, b and c are whole numbers (i.e. positive integers) . Prove that if $a = b + c$ then $a^4 + b^4 + c^4$ is double the square of a whole number.

(Folklore)

(169) 2 We are given triangle ABC. Two lines, symmetric with AC, relative to lines AB and BC are drawn, and meet at K . Prove that the line BK passes through the centre of the circumscribed circle of triangle ABC.

(V.Y. Protasov)

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(170) 3 Find all real solutions of the system of equations

$$\begin{cases} (x_3+x_4+x_5)^5 = 3x_1 \\ (x_4+x_5+x_1)^5 = 3x_2 \\ (x_5+x_1+x_2)^5 = 3x_3 \\ (x_1+x_2+x_3)^5 = 3x_4 \\ (x_2+x_3+x_4)^5 = 3x_5 \end{cases}$$

(L. Tumescu , Romania)

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(171) 4 We have a set of weights with masses 1 gm, 2 gm, 4 gm and so on, all values being powers of 2 . Some of these weights may have equal mass. Some weights were put on both sides of a balance beam, resulting in equilibrium. It is known that on the left hand side all weights were distinct . Prove that on the right hand side there were no fewer weights than on the left hand side.

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(172) 5 Is it possible to cover a plane with circles in such a way that exactly 1988 circles pass through each point?

(N . Vasiliev)

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(173) 6 The first quadrant of the Cartesian $0-x-y$ plane can be considered to be divided into an infinite set of squares of unit side length, arranged in rows and columns , formed by the axes and lines $x=i$ and $y=j$, where i and j are non-negative integers. Is it possible to write a natural number $(1, 2, 3, ...)$ in each square , so that each row and column contains each natural number exactly once?

(V . S . Shevelev)

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(174) 7 Consider a sequence of words each consisting of two letters, A and B . The first word is "A" , while the second word is "B" . The k-th word is obtained from the ($k-2$)-nd by writing after it the ($k-1$)th one. (So the first few elements of the sequence are "A" , "B" ,"AB" , "BAB" , "$ABBAB$" .) Does there exist in this sequence a "periodical" word, i.e. a word of the form $PPP...P$, where P is a word , repeated at least once?

(Remark: For instance, the word $BABBBABB$ is of the form PP , in which P is repeated exactly once .)

(A. Andjans, Riga)

– Senior

– Training

(175) 1 Is it possible to select two natural numbers m and n so that the number n results from a permutation of the digits of m, and $m+n=999...9$?

(176) 2 Two isosceles trapezoids are inscribed in a circle in such a way that each side of each trapezoid is parallel to a certain side of the other trapezoid . Prove that the diagonals of one trapezoid are equal to the diagonals of the other.

(177) 3 The set of all 10-digit numbers may be represented as a union of two subsets: the subset M consisting of all 10-digit numbers, each of which may be represented as a product of two 5-digit numbers, and the subset N , containing the remaining 10-digit numbers . Which of the sets M and N contains more elements?

(S. Fomin , Leningrad)

(178) 4 Pawns are placed on an infinite chess board so that they form an infinite square net (along any row or column containing pawns ther is a pawn , three free squares , pawn , three squares, and so on , with only every fourth row and every fourth column containing pawns). Prove that it is not possible for a knight to tour every free square once and only once.

(An old problem of A . K . Tolpugo)

– Main

(179) 1 Determine the ratio of the bases (parallel sides) of the trapezoid for which there exists a line with 6 points of intersection with the diagonals, lateral sides and extended bases cut 5 equal segments?

(E . G . Gotman)

2 same as Junior 6 (173)

(180) 3 It is known that 1 and 2 are roots of a polynomial with integer coefficients. Prove that the polynomial has a coefficient with value less than -1 .

(181) 4 There is a set of cards with numbers from 1 to 30 (which may be repeated) . Each student takes one such card. The teacher can perform the following operation: He reads a list of such numbers (possibly only one) and then asks the students to raise an arm if their number was in this list. How many times must he perform such an operation in order to determine the number on each student 's card? (Indicate the number of operations and prove that it is minimal . Note that there are not necessarily 30 students.)

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(182) 5 A $20 \times 20 \times 20$ cube is composed of 2000 bricks of size $2 \times 2 \times 1$. Prove that it is possible to pierce the cube with a needle so that the needle passes through the cube without passing through a brick .

(A . Andjans , Riga)

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(183) 6 Consider a sequence of words , consisting of the letters A and B .
The first word in the sequence is "A" . The k-th word i s obtained from the $(k-1)$-th by means of the following transformation : each A is substituted by AAB , and each B is substituted by A. It is easily seen that every word is an initial part of the next word. The initial parts of these words coincide to give a sequence of letters $AABAABAAABAABAAB...$
(a) In which place of this sequence is the 1000-th letter A?
(b) Prove that this sequence is not periodic.

(V . Galperin , Moscows)

– Autumn 1988

– Junior

– Training

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(184) 1 It is known that the proportion of people with fair hair among people with blue eyes is more than the proportion of people with fair hair among all people. Which is greater , the proportion of people with blue eyes among people with fair hair, or the proportion of people with blue eyes among all people?

(Folklore)

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(185) 2 In a triangle two altitudes are not smaller than the sides on to which they are dropped. Find the angles of the triangle.

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(186) 3 Prove that from any set of seven natural numbers (not necessarily consecutive) one can choose three, the sum of which is divisible by three.

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(187) 4 Each face of a cube has been divided into four equal quarters and each quarter is painted with one of three available colours. Quarters with common sides are painted with different colours . Prove that each of the available colours was used in painting 8 quarters.

– Main

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(188) 1 One of the numbers 1 or -1 is assigned to each vertex of a cube. To each face of the cube is assigned the integer which is the product of the four integers at the vertices of the face. Is it possible that the sum of the 14 assigned integers is 0?

(G. Galperin)

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(189) 2 A point M is chosen inside the square $ABCD$ in such a way that $\angle MAC = \angle MCD = x$. Find $\angle ABM$.

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(190) 3 Let $a_1, a_2, ..., a_n$ be an arrangement of the integers $1, 2, ..., n$. Let

$$S = \frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + ... + \frac{a_n}{1}.$$

Find a natural number n such that among the values of S for all arrangements $a_1, a_2, ..., a_n$, all the integers from n to $n + 100$ appear .

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(191) 4 (a) Two identical cogwheels with 14 teeth each are given . One is laid horizontally on top of the other in such a way that their teeth coincide (thus the projections of the teeth on the horizontal plane are identical) . Four pairs of coinciding teeth are cut off. Is it always possible to rotate the two cogwheels with respect to each other so that their common projection looks like that of an entire cogwheel?
(The cogwheels may be rotated about their common axis, but not turned over.)

(b) Answer the same question , but with two 13-tooth cogwheels and four pairs of cut-off teeth.

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(192) 5 A convex n-vertex polygon is partitioned into triangles by nonintersecting diagonals . The following operation, called perestroyka (=reconstruction) , is allowed: two triangles ABD and BCD with a common side may be replaced by the triangles ABC and ACD. By $P(n)$ denote the smallest number of perestroykas needed to transform any partitioning into any other one. Prove that
(a) $P(n) \geq n - 3$
(b) $P(n) \leq 2n - 7$
(c) $P(n) \leq 2n - 10$ if $n \geq 13$.

(D.Fomin , based on ideas of W. Thurston , D . Sleator, R. Tarjan)

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(193) 6 Does there exist a natural number which is not a divisor of any natural number whose decimal expression consists of zeros and ones, with no more than 1988 ones?

– Senior

– Training

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(194) 1 Is there a power of 2 such that it is possible to rearrange the digits, giving another power of 2?

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(195) 2 Let N be the orthocentre of triangle ABC (i .e. the point where the altitudes meet). Prove that the circumscribed circles of triangles ABN, ACN and BCN each have equal radius.

(196) 3 Prove that for each vertex of a polyhedron it is possible to attach a natural number so that for each pair of vertices with a common edge, the attached numbers are not relatively prime (i.e. they have common divisors), and with each pair of vertices without a common edge the attached numbers are relatively prime.

(Note: there are infinitely many prime numbers.)

(197) 4 A page of an exercise book is painted with 23 colours, arranged in squares. A pair of colours is called *good* if there are neighbouring squares painted with these colours. What is the minimum number of good pairs?

– Main

(198) 1 What is the smallest number of squares of a chess board that can be marked in such a manner that
(a) no two marked squares may have a common side or a common vertex, and
(b) any unmarked square has a common side or a common vertex with at least one marked square?
Indicate a specific configuration of marked squares satisfying (a) and (b) and show that a lesser number of marked squares will not suffice.

(A. Andjans, Riga)

(199) 2 Prove that $a^2pq + b^2qr + c^2rp \le 0$, whenever a, b and c are the lengths of the sides of a triangle and $p + q + r = 0$.

(J. Mustafaev , year 12 student, Baku)

(200) 3 The integers $1, 2, ..., n$ are rearranged in such a way that if the integer $k, 1 \le k \le n$, is not the first term, then one of the integers $k+1$ or $k-1$ occurs to the left of k . How many arrangements of the integers $1, 2, ..., n$ satisfy this condition?

(A. Andjans, Riga)

(201) 4 There are 1988 towns and 4000 roads in a certain country (each road connects two towns) . Prove that there is a closed path passing through no more than 20 towns.

(A. Razborov , Moscow)

5 same as Junior 6 (193)

(202) 6 M is an interior point of a rectangle $ABCD$ and S is its area.
Prove that $S \leq AM \cdot CM + BM \cdot DM$.

(I.J . Goldsheyd)