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– Day 1

1 The positive integers $a_0, a_1, a_2, \ldots, a_{3030}$ satisfy

$$2a_{n+2} = a_{n+1} + 4a_n \text{ for } n = 0, 1, 2, \ldots, 3028.$$

Prove that at least one of the numbers $a_0, a_1, a_2, \ldots, a_{3030}$ is divisible by 2^{2020}.

2 Find all lists $(x_1, x_2, \ldots, x_{2020})$ of non-negative real numbers such that the following three conditions are all satisfied:

- $x_1 \le x_2 \le \ldots \le x_{2020}$;
- $x_{2020} \le x_1 + 1$;
- there is a permutation $(y_1, y_2, \ldots, y_{2020})$ of $(x_1, x_2, \ldots, x_{2020})$ such that

$$\sum_{i=1}^{2020}((x_i+1)(y_i+1))^2 = 8\sum_{i=1}^{2020} x_i^3.$$

[i]A permutation of a list is a list of the same length, with the same entries, but the entries are allowed to be in any order. For example, $(2, 1, 2)$ is a permutation of $(1, 2, 2)$, and they are both permutations of $(2, 2, 1)$. Note that any list is a permutation of itself.[/i]

3 Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$ and $\angle B = \angle D = \angle F$ and the (interior) angle bisectors of $\angle A,\ \angle C$, and $\angle E$ are concurrent.

Prove that the (interior) angle bisectors of $\angle B,\ \angle D$, and $\angle F$ must also be concurrent.

[i]Note that $\angle A = \angle FAB$. The other interior angles of the hexagon are similarly described.[/i]

– Day 2

4 A permutation of the integers $1, 2, \ldots, m$ is called *fresh* if there exists no positive integer $k < m$ such that the first k numbers in the permutation are $1, 2, \ldots, k$ in some order. Let f_m be the number of fresh permutations of the integers $1, 2, \ldots, m$.

Prove that $f_n \ge n \cdot f_{n-1}$ for all $n \ge 3$.

[i]For example, if $m = 4$, then the permutation $(3, 1, 4, 2)$ is fresh, whereas the permutation $(2, 3, 1, 4)$ is not.[/i]

5 Consider the triangle ABC with $\angle BCA > 90^\circ$. The circumcircle Γ of ABC has radius R. There is a point P in the interior of the line segment AB such that $PB = PC$ and the length of PA is R. The perpendicular bisector of PB intersects Γ at the points D and E.

Prove P is the incentre of triangle CDE.

6 Let $m > 1$ be an integer. A sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = a_2 = 1$, $a_3 = 4$, and for all $n \geq 4$,

$$a_n = m(a_{n-1} + a_{n-2}) - a_{n-3}.$$

Determine all integers m such that every term of the sequence is a square.