Tournament Of Towns 1999

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by parmenides51

–	Spring 1999
–	Junior O Level
1	A father and his son are skating around a circular skating rink. From time to time, the father overtakes the son. After the son starts skating in the opposite direction, they begin to meet five times more often. What is the ratio of the skating speeds of the father and the son? (Tairova)
2	ABC is a right-angled triangle. A square $ABDE$ is constructed on the opposite side of the hypothenuse AB from C. The bisector of $\angle C$ cuts DE at F. If $AC = 1$ and $BC = 3$, compute $\frac{DF}{EF}$. (A Blinkov)
3	Several positive integers $a_0, a_1, a_2, ..., a_n$ are written on a board. On a second board, we write the amount b_0 of numbers written on the first board, the amount b_1 of numbers on the first board exceeding 1, the amount b_2 of numbers greater than 2, and so on as long as the bs are still positive. Then we stop, so that we do not write any zeros. On a third board we write the numbers $c_0, c_1, c_2, ...$ using the same rules as before, but applied to the numbers $b_0, b_1, b_2, ...$ of the second board. Prove that the same numbers are written on the first and the third boards. (H. Lebesgue - A Kanel)
4	A black unit equilateral triangle is drawn on the plane. How can we place nine tiles, each a unit equilateral triangle, on the plane so that they do not overlap, and each tile covers at least one interior point of the black triangle? (Folklore)
5	A square is cut into 100 rectangles by 9 straight lines parallel to one of the sides and 9 lines parallel to another. If exactly 9 of the rectangles are actually squares, prove that at least two of these 9 squares are of the same size . (V Proizvolov)
–	Junior A Level

1 There is 500 dollars in a bank. Two bank operations are allowed: to withdraw 300 dollars from the bank or to deposit 198 dollars into the bank. These operations can be repeated as many times as necessary but only the money that was initially in the bank can be used. What is the largest amount of money that can be borrowed from the bank? How can this be done?

(AK Tolpygo)

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2 Let O be the intersection point of the diagonals of a parallelogram $ABCD$. Prove that if the line BC is tangent to the circle passing through the points A, B, and O, then the line CD is tangent to the circle passing through the points B, C and O.

(A Zaslavskiy)

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3 Two players play the following game. The first player starts by writing either 0 or 1 and then, on his every move, chooses either 0 or 1 and writes it to the right of the existing digits until there are 1999 digits. Each time the first player puts down a digit (except the first one) , the second player chooses two digits among those already written and swaps them. Can the second player guarantee that after his last move the line of digits will be symmetrical about the middle digit?

(I Izmestiev)

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4 n diameters divide a disk into $2n$ equal sectors. n of the sectors are coloured blue , and the other n are coloured red (in arbitrary order) . Blue sectors are numbered from 1 to n in the anticlockwise direction, starting from an arbitrary blue sector, and red sectors are numbered from 1 to n in the clockwise direction, starting from an arbitrary red sector. Prove that there is a semi-disk containing sectors with all numbers from 1 to n.

(V Proizvolov)

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5 The sides AB and AC are tangent at points P and Q, respectively, to the incircle of a triangle $ABC.R$ and S are the midpoints of the sides AC and BC, respectively, and T is the intersection point of the lines PQ and RS. Prove that T lies on the bisector of the angle B of the triangle.

(M Evdokimov)

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6 A rook is allowed to move one cell either horizontally or vertically. After 64 moves the rook visited all cells of the 8×8 chessboard and returned back to the initial cell. Prove that the number of moves in the vertical direction and the number of moves in the horizontal direction cannot be equal.

(A Shapovalov, R Sadykov)

– Senior O Level

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1 In a row are written 1999 numbers such that except the first and the last , each is equal to the sum of its neighbours. If the first number is 1, find the last number.

(V Senderov)

2 same as Junior O2

3 same as Junior O3

4 same as Junior O4

5 Two people play a game on a 9×9 board. They move alternately. On each move, the first player draws a cross in an empty cell, and the second player draws a nought in an empty cell. When all 81 cells are filled, the number K of rows and columns in which there are more crosses and the number H of rows and columns in which there are more noughts are counted. The score for the first player is the difference $B = K - H$. Find a value of B such that the first player can guarantee a score of at least B, while the second player can hold the first player's score to at most B, regardless how the opponent plays.

(A Kanel)

– Senior A Level

1 A convex polyhedron is floating in a sea. Can it happen that 90% of its volume is below the water level, while more than half of its surface area is above the water level?

(A Shapovalov)

2 Let all vertices of a convex quadrilateral $ABCD$ lie on the circumference of a circle with center O. Let F be the second intersection point of the circumcircles of the triangles ABO and CDO. Prove that the circle passing through the points A, F and D also passes through the intersection point of the segments AC and BD.

(A Zaslavskiy)

3 Find all pairs (x, y) of integers satisfying the following condition:
each of the numbers $x^3 + y$ and $x + y^3$ is divisible by $x^2 + y^2$.

(S Zlobin)

4 same as Junior A4

5 For every non-negative integer i, define the number $M(i)$ as follows:
write i down as a binary number, so that we have a string of zeroes and ones, if the number of ones in this string is even, then set $M(i) = 0$, otherwise set $M(i) = 1$. (The first terms of the sequence $M(i)$, $i = 0, 1, 2, ...$ are $0, 1, 1, 0, 1, 0, 0, 1, ...$)
(a) Consider the finite sequence $M(O), M(1), ..., M(1000)$.
Prove that there are at least 320 terms in this sequence which are equal to their neighbour on the right : $M(i) = M(i + 1)$.

(b) Consider the finite sequence $M(O), M(1), ..., M(1000000)$.
Prove that the number of terms $M(i)$ such that $M(i) = M(i+7)$ is at least 450000.

(A Kanel)

6 same as Junior A6

– Autumn 1999

– Junior O Level

1 A right-angled triangle made of paper is folded along a straight line so that the vertex at the right angle coincides with one of the other vertices of the triangle and a quadrilateral is obtained .
(a) What is the ratio into which the diagonals of this quadrilateral divide each other?
(b) This quadrilateral is cut along its longest diagonal. Find the area of the smallest piece of paper thus obtained if the area of the original triangle is 1 .

(A Shapovalov)

2 Let $d = a^{1999} + b^{1999} + c^{1999}$, where a, b and c are integers such that $a + b + c = 0$.
(a) May it happen that $d = 2$?
(b) May it happen that d is prime?

(V Senderov)

3 There are n straight lines in the plane such that each intersects exactly 1999 of the others . Find all posssible values of n.

(R Zhenodarov)

4 Every 24 hours , the minute hand of an ordinary clock completes 24 revolutions while the hour hand completes 2. Every 24 hours , the minute hand of an Italian clock completes 24 revolutions while the hour hand completes only 1 . The minute hand of each clock is longer than the hour hand, and ”zero hour” is located at the top of the clock's face. How many positions of the two hands can occur on an Italian clock within a 24-hour period that are possible on an ordinary one?

(Folklore)

5 Is it possible to divide a 6×6 chessboard into 18 rectangles, each either 1×2 or 2×1, and to draw exactly one diagonal on each rectangle such that no two of these diagonals have a common endpoint?

(A Shapovalov)

– Senior O Level

1 The incentre of a triangle is joined by three segments to the three vertices of the triangle, thereby dividing it into three smaller triangles. If one of these three triangles is similar to the original triangle, find its angles.

(A Shapovalov)

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2 Prove that there exist infinitely many odd positive integers n for which the number $2^n + n$ is composite.

(V Senderov)

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3 same as Junior O3

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4 Is it possible to divide the integers from 1 to 100 inclusive into 50 pairs such that for $1 \leq k \leq 50$, the difference between the two numbers in the k-th pair is k?

(V Proizvolov)

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5 Is it possible to divide a 8×8 chessboard into 32 rectangles, each either 1×2 or 2×1, and to draw exactly one diagonal on each rectangle such that no two of these diagonals have a common endpoint?

(A Shapovalov)

– Senior A Level

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1 For what values o f n is it possible to place the integers from 1 to n inclusive on a circle (not necessarily in order) so that the sum of any two successive integers in the circle is divisible by the next one in the clockwise order?

(A Shapovalov)

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2 On a rectangular piece of paper there are
(a) several marked points all on one straight line,
(b) three marked points (not necessarily on a straight line).
We are allowed to fold the paper several times along a straight line not containing marked points and then puncture the folded paper with a needle. Show that this can be done so that after the paper has been unfolded all the marked points are punctured and there are no extra holes.

(A Shapovalov)

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3 Tireless Thomas and Jeremy construct a sequence. At the beginning there is one positive integer in the sequence. Then they successively write new numbers in the sequence in the following way: Thomas obtains the next number by adding to the previous number one of its (decimal) digits, while Jeremy obtains the next number by subtracting from the previous number one of its digits. Prove that there is a number in this sequence which will be repeated at least 100 times.

(A Shapovalov)

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4 Points K, L on sides AC, CB respectively of a triangle ABC are the points of contact of the excircles with the corresponding sides . Prove that the straight line through the midpoints of KL and AB
(a) divides the perimeter of triangle ABC in half,
(b) is parallel to the bisector of angle ACB.

(L Emelianov)

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5 (a) The numbers $1, 2, ..., 100$ are divided into two groups so that the sum of all numbers in one group is equal to that in the other. Prove that one can remove two numbers from each group so that the sums of all numbers in each group are still the same.
(b) The numbers $1, 2, ..., n$ are divided into two groups so that the sum of all numbers in one group is equal to that in the other . Is it true that for every such$n > 4$ one can remove two numbers from each group so that the sums of all numbers in each group are still the same?

(A Shapovalov)

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6 On a large chessboard $2n$ of its 1×1 squares have been marked such thar the rook (which moves only horizontally or vertically) can visit all the marked squares without jumpin over any unmarked ones. Prove that the figure consisting of all the marked squares can be cut into rectangles.

(A Shapovalov)

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7 Prove that any convex polyhedron with $10n$ faces, has at least n faces with the same number of sides.

(A Kanel)
