Olympic Revenge 2002

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1 Show that there is no function $f:\mathbb{N}^*\to\mathbb{N}^*$ such that $f^n(n)=n+1$ for all n (when f^n is the nth iteration of f)

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2 $ABCD$ is a inscribed quadrilateral. P is the intersection point of its diagonals. O is its circumcenter. Γ is the circumcircle of ABO. Δ is the circumcircle of CDO. M is the midpoint of arc AB on Γ who doesn't contain O. N is the midpoint of arc CD on Δ who doesn't contain O.

Show that M,N,P are collinear.

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3 Show that if x,y,z,w are positive reals, then

$$\frac{3}{2}\sqrt{(x^2+y^2)(w^2+z^2)}+\sqrt{(x^2+w^2)(y^2+z^2)-3xyzw}\geq(x+z)(y+w)$$

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4 Find all pairs (m,n) of positive integers such that there exists a polyhedron, with all faces being regular polygons, such that each vertex of the polyhedron is the vertex of exactly three faces, two of them having m sides, and the other having n sides.

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5 In a "Hanger Party", the guests are initially dressed. In certain moments, the host chooses a guest, and the chosen guest and all his friends will wear its respective clothes if they are naked, and undress it if they are dressed.

It is possible that, in some moment, the guests are naked, independent of their mutual friendships? (Suppose friendship is reciprocal.)

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6 Let p a prime number, and N the number of matrices $p\times p$

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \dots & a_{pp} \end{matrix}$$

such that $a_{ij}\in\{0,1,2,\dots,p\}$ and if $i\leq i'$ and $j\leq j'$, then $a_{ij}\leq a_{i'j'}$.

Find $N \pmod{p}$.

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7 Show that

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

is an integer, for any positive integer n.