Olympic Revenge 2007

www.artofproblemsolving.com/community/c1134214

by cyshine

1 Let a, b, n be positive integers with $a, b > 1$ and $\gcd(a, b) = 1$. Prove that n divides $\phi\left(a^n + b^n\right)$.

---

2 Let $a, b, c \in \mathbb{R}$ with $abc = 1$. Prove that

$$a^2 + b^2 + c^2 + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + 2\left(a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 6 + 2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c} + \frac{c}{a} + \frac{c}{b} + \frac{b}{c}\right)$$

---

3 The triangles BCD and ACE are externally constructed to sides BC and CA of a triangle ABC such that $AE = BD$ and $\angle BDC + \angle AEC = 180^\circ$. Let F be a point on segment AB such that $\frac{AF}{FB} = \frac{CD}{CE}$. Prove that $\frac{DE}{CD+CE} = \frac{EF}{BC} = \frac{FD}{AC}$.

---

4 Let $A_1A_2B_1B_2$ be a convex quadrilateral. At adjacent vertices A_1 and A_2 there are two Argentinian cities. At adjacent vertices B_1 and B_2 there are two Brazilian cities. There are a Argentinian cities and b Brazilian cities in the quadrilateral interior, no three of which collinear. Determine if it's possible, independently from the cities position, to build straight roads, each of which connects two Argentinian cities ou two Brazilian cities, such that:
• Two roads does not intersect in a point which is not a city; • It's possible to reach any Argentinian city from any Argentinian city using the roads; and • It's possible to reach any Brazilian city from any Brazilian city using the roads.

If it's always possible, construct an algorithm that builds a possible set of roads.

---

5 Find all functions $f\colon R \to R$ such that

$$f\left(x^2 + yf(x)\right) = f(x)^2 + xf(y)$$

for all reals x, y.

---

6 *Mediovagio* is a computer game that consists in a 3×3 table in which each of the nine cells has a integer number from 1 to n. When one clicks a cell, the numbers in the clicked cell and in the cells that share an edge with it are increased by 1 and the sum is evaluated $\bmod n$. Determine the values of n for which it's possible, with a finite number of clicks, obtain any combination of numbers from an given initial combination.

EDIT: I corrected the statement.
