Olympic Revenge 2003

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1 Let ABC be a triangle with circumcircle Γ. D is the midpoint of arc BC (this arc does not contain A). E is the common point of BC and the perpendicular bisector of BD. F is the common point of AC and the parallel to AB containing D. G is the common point of EF and AB. H is the common point of GD and AC. Show that GAH is isosceles.

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2 Let x_n the sequence defined by any nonnegatine integer x_0 and $x_{n+1} = 1 + \prod_{0 \leq i \leq n} x_i$
Show that there exists prime p such that $p \not| x_n$ for any n.

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3 Let ABC be a triangle with $\angle BAC = 60°$. A' is the symmetric point of A wrt $\overline{BC}$. D is the point in $\overline{AC}$ such that $\overline{AB} = \overline{AD}$. H is the orthocenter of triangle ABC. l is the external angle bisector of $\angle BAC$. $\{M\} = \overline{A'D} \cap l$,$\{N\} = \overline{CH} \cap l$. Show that $\overline{AM} = \overline{AN}$.

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4 In the Mobius Planet (a plane and infinite planet!, in a similar manner to the $N \times N$ lattice), the Supreme King Mobius is planning to construct a water reservoir. There are some restrictions to this project:
1. There exists only $k < \infty$ bricks.
2. These bricks will delimit a closed finite area.
What is the maximum area of this resevoir in function of k?

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5 Let $[n] = \{1, 2, ..., n\}$.Let p be any prime number.
Find how many finite non-empty sets $S \in [p] \times [p]$ are such that

$$p | \sum_{(x,y) \in S} x, p | \sum_{(x,y) \in S} y$$

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6 Find all functions $f : R^* \to R$ such that $f(x) \neq x$ and

$$f(y(f(x) - x)) = \frac{f(x)}{y} - \frac{f(y)}{x}$$

for any $x, y \neq 0$.

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7 Let X be a subset of R_+^* with m elements.
Find X such that the number of subsets with the same sum is maximum.
