Olympic Revenge 2004

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1 ABC is a triangle and D is an internal point such that $\angle DAB = \angle DBC = \angle DCA$. O_a is the circumcenter of DBC. O_b is the circumcenter of DAC. O_c is the circumcenter of DAB. Show that if the area of ABC and $O_aO_bO_c$ are equal then ABC is equilateral.

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2 If a, b, c, x are positive reals, show that

$$\frac{a^{x+2}+1}{a^x bc+1}+\frac{b^{x+2}+1}{b^x ac+1}+\frac{c^{x+2}+1}{c^x ab+1}\geq 3$$

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3 ABC is a triangle and ω its incircle. Let P, Q, R be the intersections with ω and the sides BC, CA, AB respectively. AP cuts ω in P and X. BX, CX cut ω in M, N respectively. Show that MR, NQ, AP are parallel or concurrent.

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4 Find all functions $f : R \to R$ such that for any reals x, y, $f(x^2+y) = f(x)f(x+1)+f(y)+2x^2y$.

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5 $a_0 = a_1 = 1$ and $a_{n+1}.a_{n-1} = a_n.(a_n + 1)$ for all positive integers n.

prove that a_n is one integer for all positive integers n.

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6 For any natural n, $f(n)$ is the number of labeled digraphs with n vertices such that for any vertex the number if in-edges is equal to the number of out-edges and the total of (in+out) edges is even. Let $g(n)$ be the odd-analogous of $f(n)$. Find $g(n) - f(n)$ with proof .

Dado n natural, seja $f(n)$ o nmero de grafos rotulados direcionados com n vrtices de modo que em cada vrtice o nmero de arestas que chegam igual ao nmero de
arestas que saem e o nmero de arestas total do grafo par . Defina $g(n)$ analogamente trocando ”par” por ”mpar” na definio acima. Calcule $f(n) - g(n)$.

(Observao: Um grafo rotulado direcionado um par $G = (V, E)$ onde $V = \{1, 2, , n\}$ e E um subconjunto de $V^2 - \{(i, i); 0 < i < n + 1\}$).
