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– Day 1

1 Let q be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard:

-In the first line, Gugu writes down every number of the form $a-b$, where a and b are two (not necessarily distinct) numbers on his napkin.
-In the second line, Gugu writes down every number of the form qab, where a and b are two (not necessarily distinct) numbers from the first line.
-In the third line, Gugu writes down every number of the form $a^2+b^2-c^2-d^2$, where a, b, c, d are four (not necessarily distinct) numbers from the first line.

Determine all values of q such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

2 Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

3 Consider the set of all integer points in Z^3. Sasha and Masha play such a game. At first, Masha marks an arbitrary point. After that, Sasha marks all the points on some a plane perpendicular to one of the coordinate axes and at no point, which Masha noted. Next, they continue to take turns (Masha can't to select previously marked points, Sasha cannot choose the planes on which there are points said Masha). Masha wants to mark n consecutive points on some line that parallel to one of the coordinate axes, and Sasha seeks to interfere with it. Find all n, in which Masha can achieve the desired result.

– Day 2

4 Let n be an odd integer. Consider a square lattice of size $n \times n$, consisting of n^2 unit squares and $2n(n+1)$ edges. All edges are painted in red or blue so that the number of red edges does not exceed n^2. Prove that there is a cell that has at least three blue edges.

5 Find the smallest positive number λ such that for an arbitrary 12 points on the plane $P_1, P_2, ...P_{12}$ (points may coincide), with distance between arbitrary two of them does not exceeds 1, holds the inequality $\sum_{1 \le i \le j \le 12} P_iP_j^2 \le \lambda$

6 Find the smallest positive integer n or show no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers $(a_1, a_2, \ldots, a_n)$ such that both

$$a_1 + a_2 + \cdots + a_n \quad \text{and} \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

are integers.

– Day 3

7 The prime number $p > 2$ and the integer n are given. Prove that the number pn^2 has no more than one divisor d for which $n^2 + d$ is the square of the natural number.
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8 A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation

$$a_n = -\max_{i+j=n}(a_i + a_j) \qquad \text{for all} \quad n > 2017.$$

Prove that the sequence is bounded, i.e., there is a constant M such that $|a_n| \leq M$ for all positive integers n.

9 Let AA_1, BB_1, CC_1 be the heights of triangle ABC and H be its orthocenter. Liune ℓ parallel to AC, intersects straight lines AA_1 and CC_1 at points A_2 and C_2, respectively. Suppose that point B_1 lies outside the circumscribed circle of triangle A_2HC_2. Let B_1P and B_1T be tangent to of this circle. Prove that points A_1, C_1, P, and T are cyclic.

– Day 4

10 Let ABC be a triangle with AH altitude. The point K is chosen on the segment AH as follows such that $AH = 3KH$. Let O be the center of the circle circumscribed around by triangle ABC, M and N be the midpoints of AC and AB respectively. Lines KO and MN intersect at the point Z, a perpendicular to OK passing through point Z intersects lines AC and AB at points X and Y respectively. Prove that $\angle XKY = \angle CKB$.

11 $2n$ students take part in a math competition. First, each of the students sends its task to the members of the jury, after which each of the students receives from the jury one of proposed tasks (all received tasks are different). Let's call the competition *honest*, if there are n students who were given the tasks suggested by the remaining n participants. Prove that the number of task distributions in which the competition is honest is a square of natural numbers.

12 Let n be a positive integer and $a_1, a_2, \ldots, a_n$ be integers. Function $f : \mathbb{Z} \to \mathbb{R}$ is such that for all integers k and l, $l \neq 0$,

$$\sum_{i=1}^{n} f(k + a_i l) = 0.$$

Prove that $f \equiv 0$.