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by parmenides51, hajimbrak

– Day 1

1 Let O be the circumcenter of the triangle ABC, A' be a point symmetric of A wrt line BC, X is an arbitrary point on the ray AA' ($X \neq A$). Angle bisector of angle BAC intersects the circumcircle of triangle ABC at point D ($D \neq A$). Let M be the midpoint of the segment DX. A line passing through point O parallel to AD, intersects DX at point N. Prove that angles BAM and CAN angles are equal.

2 In a football tournament, n teams play one round ($n \vdots 2$). In each round should play $n/2$ pairs of teams that have not yet played. Schedule of each round takes place before its holding. For which smallest natural k such that the following situation is possible:
after k tours, making a schedule of $k+1$ rounds already is not possible,
i.e. these n teams cannot be divided into $n/2$ pairs, in each of which there are teams that have not played in the previous k rounds.

PS. The 3 vertical dots notation in the first row, I do not know what it means.

3 Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

Proposed by Belgium

– Day 2

4 A prime number $p > 3$ is given. Prove that integers less than p, it is possible to divide them into two non-empty sets such that the sum of the numbers in the first set will be congruent modulo p to the product of the numbers in the second set.

5 For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \le i \le n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i-th step he chooses x_i among the remaining numbers so as to minimise the value of

$|x_1 + x_2 + \cdots x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \leq cD$.

Proposed by Georgia

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6 Given an acute triangle ABC, H is the foot of the altitude drawn from the point A on the line BC, P and $K \neq H$ are arbitrary points on the segments AH andBC respectively. Segments AC and BP intersect at point B_1, lines AB and CP at point C_1. Let X and Y be the projections of point H on the lines KB_1 and KC_1, respectively. Prove that points A, P, X and Y lie on one circle.

– Day 3

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7 Let A and B be two sets of real numbers. Suppose that the elements of the set $AB = \{ab : a \in A, b \in B\}$ form a finite arithmetic progression. Prove that one of these sets contains no more than three elements

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8 Find all functions $f : R \to R$ such that $f(x)f(yf(x) - 1) = x^2 f(y) - f(x)$ for all real x, y

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9 The set M consists of n points on the plane and satisfies the conditions: • there are 7 points in the set M, which are vertices of a convex heptagon, • for arbitrary five points with M, which are vertices of a convex pentagon, there is a point that also belongs to M and lies inside this pentagon.
Find the smallest possible value that n can take .

– Day 4

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10 Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

Proposed by Titu Andreescu, USA

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11 Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$.
(Here we always assume that an angle bisector is a ray.)

Proposed by Sergey Berlov, Russia

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12 For a given natural n, we consider the set $A \subset \{1, 2, ..., n\}$, which consists of at least $\left[\frac{n+1}{2}\right]$ items. Prove that for $n \geq 2015$ the set A contains a three-element arithmetic sequence.