Canadian Mathematical Olympiad Qualification Repechage 2020

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by parmenides51

1 Show that for all integers $a \geq 1$,$\lfloor\sqrt{a}+\sqrt{a+1}+\sqrt{a+2}\rfloor=\lfloor\sqrt{9a+8}\rfloor$

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2 Given a set S, of integers, an *optimal partition* of S into sets T, U is a partition which minimizes the value $|t-u|$, where t and u are the sum of the elements of T and U respectively.
Let P be a set of distinct positive integers such that the sum of the elements of P is $2k$ for a positive integer k, and no subset of P sums to k.
Either show that there exists such a P with at least 2020 different optimal partitions, or show that such a P does not exist.

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3 Let N be a positive integer and $A=a_1,a_2,...,a_N$ be a sequence of real numbers.
Define the sequence $f(A)$ to be

$$f(A)=\left(\frac{a_1+a_2}{2},\frac{a_2+a_3}{2},...,\frac{a_{N-1}+a_N}{2},\frac{a_N+a_1}{2}\right)$$

and for k a positive integer define $f^k(A)$ to be f applied to A consecutively k times (i.e. $f(f(...f(A)))$)
Find all sequences $A=(a_1,a_2,...,a_N)$ of integers such that $f^k(A)$ contains only integers for all k.

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4 Determine all graphs G with the following two properties: • G contains at least one Hamilton path. • For any pair of vertices, $u,v \in G$, if there is a Hamilton path from u to v then the edge uv is in the graph G

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5 We define the following sequences:
Sequence A has $a_n=n$.
Sequence B has $b_n=a_n$ when $a_n\not\equiv 0$ (mod 3) and $b_n=0$ otherwise.
Sequence C has $c_n=\sum_{i=1}^{n} b_i$
. Sequence D has $d_n=c_n$ when $c_n\not\equiv 0$ (mod 3) and $d_n=0$ otherwise.
Sequence E has $e_n=\sum_{i=1}^{n} d_i$
Prove that the terms of sequence E are exactly the perfect cubes.

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6 In convex pentagon $ABCDE, AC$ is parallel to DE, AB is perpendicular to AE, and BC is perpendicular to CD. If H is the orthocentre of triangle ABC and M is the midpoint of segment DE, prove that AD, CE and HM are concurrent.

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7 Let a,b,c be positive real numbers with $ab+bc+ac=abc$. Prove that

$$\frac{bc}{a^{a+1}}+\frac{ac}{b^{b+1}}+\frac{ab}{c^{c+1}}\geq\frac{1}{3}$$

8 Find all pairs (a, b) of positive rational numbers such that $\sqrt[b]{a} = ab$