Austrian-Polish Competition 1980

www.artofproblemsolving.com/community/c1135989

by parmenides51, orl, Kezer

–	Individual
1	Given three infinite arithmetic progressions of natural numbers such that each of the numbers 1,2,3,4,5,6,7 and 8 belongs to at least one of them, prove that the number 1980 also belongs to at least one of them.
2	A sequence of integers $1 = x_1 < x_2 < x_3 <$ ... satisfies $x_{n+1} \leq 2n$ for all n. Show that every positive integer k can be written as $x_j - x_i$ for some i, j.
3	Prove that the sum of the six angles subtended at an interior point of a tetrahedron by its six edges is greater than 540.
4	Prove that $\sum \frac{1}{i_1 i_2 \ldots i_k} = n$ is taken over all non-empty subsets $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$. (The k is not fixed, so we are summing over all the $2^n - 1$ possible nonempty subsets.)
5	Let $A_1A_2A_3$ be a triangle and, for $1 \leq i \leq 3$, let B_i be an interior point of edge opposite A_i. Prove that the perpendicular bisectors of A_iB_i for $1 \leq i \leq 3$ are not concurrent.
6	Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers satisfying the inequality $$\lvert a_{k+m} - a_k - a_m \rvert \leq 1 \quad \text{for all } k, m \in \mathbb{Z}_{>0}.$$ Show that the following inequality holds for all positive integers k, m $$\left\lvert \frac{a_k}{k} - \frac{a_m}{m} \right\rvert < \frac{1}{k} + \frac{1}{m}.$$
–	Team
7	Find the greatest natural number n such there exist natural numbers $x_1, x_2, \ldots, x_n$ and natural $a_1 < a_2 < \ldots < a_{n-1}$ satisfying the following equations for $i = 1, 2, \ldots, n-1$: $$x_1 x_2 \ldots x_n = 1980 \quad \text{and} \quad x_i + \frac{1980}{x_i} = a_i.$$

8 Let S be a set of 1980 points in the plane such that the distance between every pair of them is at least 1. Prove that S has a subset of 220 points such that the distance between every pair of them is at least $\sqrt{3}$.

---

9 Through the endpoints A and B of a diameter AB of a given circle, the tangents ℓ and m have been drawn. Let $C \neq A$ be a point on ℓ and let q_1, q_2 be two rays from C. Ray q_i cuts the circle in D_i and E_i with D_i between C and $E_i, i = 1, 2$. Rays AD_1, AD_2, AE_1, AE_2 meet m in the respective points M_1, M_2, N_1, N_2. Prove that $M_1M_2 = N_1N_2$.
