Austrian-Polish Competition 2003

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–	Individual
1	Find all real polynomials $p(x)$ such that $p(x-1)p(x+1) = p(x^2-1)$.
2	The sequence $a_0, a_1, a_2, ..$ is defined by $a_0 = a, a_{n+1} = a_n + L(a_n)$, where $L(m)$ is the last digit of m (eg $L(14) = 4$). Suppose that the sequence is strictly increasing. Show that infinitely many terms must be divisible by $d = 3$. For what other d is this true?
3	ABC is a triangle. Take $a = BC$ etc as usual. Take points T_1, T_2 on the side AB so that $AT_1 = T_1T_2 = T_2B$. Similarly, take points T_3, T_4 on the side BC so that $BT_3 = T_3T_4 = T_4C$, and points T_5, T_6 on the side CA so that $CT_5 = T_5T_6 = T_6A$. Show that if $a' = BT_5, b' = CT_1, c' = AT_3$, then there is a triangle $A'B'C'$ with sides a', b', c' ($a' = B'C'$ etc). In the same way we take points T'_i on the sides of $A'B'C'$ and put $a'' = B'T'_6, b'' = C'T'_2, c'' = A'T'_4$. Show that there is a triangle $A''B''C''$ with sides $a''b'', c''$ and that it is similar to ABC. Find a''/a. Austrian-Polish 2003
5	A triangle with sides a, b, c has area S. The distances of its entroid from the vertices are x, y, z. Show that, if $$(x+y+z)^2 \le (a^2+b^2+c^2)/2 + 2S\sqrt{3},$$ then the triangle is equilateral. posted for the latex, this hide shall be removed soon, problem comes from Austrian Polish 2003
4	A positive integer m is alpine if m divides $2^{2n+1} + 1$ for some positive integer n. Show that the product of two alpine numbers is alpine.
6	$ABCD$ is a tetrahedron such that we can find a sphere $k(A, B, C)$ through A, B, C which meets the plane BCD in the circle diameter BC, meets the plane ACD in the circle diameter AC, and meets the plane ABD in the circle diameter AB. Show that there exist spheres $k(A, B, D)$, $k(B, C, D)$ and $k(C, A, D)$ with analogous properties. Austrian-Polish 2003
–	Team

7 Put $f(n) = \frac{n^n-1}{n-1}$. Show that $n!^{f(n)}$ divides $(n^n)!$.
Find as many positive integers as possible for which $n!^{f(n)+1}$ does not divide $(n^n)!$.

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9 Take any 26 distinct numbers from 1, 2, ... , 100. Show that there must be a non-empty subset of the 26 whose product is a square.

I think that the upper limit for such subset is 37.

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8 Given reals $x_1 \geq x_2 \geq ... \geq x_{2003} \geq 0$, show that

$$x_1^n - x_2^n + x_2^n - ... - x_{2002}^n + x_{2003}^n \geq (x_1 - x_2 + x_3 - x_4 + ... - x_{2002} + x_{2003})^n$$

for any positive integer n.

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10 What is the smallest number of 5×1 tiles which must be placed on a 31×5 rectangle (each covering exactly 5 unit squares) so that no further tiles can be placed? How many different ways are there of placing the minimal number (so that further tiles are blocked)? What are the answers for a 52×5 rectangle?
