Austrian-Polish Competition 1981

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–	Individual
1	Find the smallest n for which we can find 15 distinct elements $a_1, a_2, ..., a_{15}$ of $\{16, 17, ..., n\}$ such that a_k is a multiple of k.
2	The sequence $a_0, a_1, a_2, ...$ is defined by $a_{n+1} = a_n^2 + (a_n - 1)^2$ for $n \geq 0$. Find all rational numbers a_0 for which there exist four distinct indices k, m, p, q such that $a_q - a_p = a_m - a_k$.
3	Given is a triangle ABC, the inscribed circle G of which has radius r. Let r_a be the radius of the circle touching AB, AC and G. [This circle lies inside triangle ABC.] Define r_b and r_c similarly. Prove that $r_a + r_b + r_c \geq r$ and find all cases in which equality occurs. *Bosnia - Herzegovina Mathematical Olympiad 2002*
4	Let $n \geq 3$ cells be arranged into a circle. Each cell can be occupied by 0 or 1. The following operation is admissible: Choose any cell C occupied by a 1, change it into a 0 and simultaneously reverse the entries in the two cells adjacent to C (so that x, y become $1 - x$, $1 - y$). Initially, there is a 1 in one cell and zeros elsewhere. For which values of n is it possible to obtain zeros in all cells in a finite number of admissible steps?
5	Let $P(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ be a polynomial with rational coefficients. Show that if $P(x)$ has exactly one real root ξ, then ξ is a rational number.
6	The sequences $(x_n), (y_n), (z_n)$ are given by $x_{n+1} = y_n + \frac{1}{x_n}$,$y_{n+1} = z_n + \frac{1}{y_n}$,$z_{n+1} = x_n + \frac{1}{z_n}$ for $n \geq 0$ where x_0, y_0, z_0 are given positive numbers. Prove that these sequences are unbounded.
–	Team
7	Let $a > 3$ be an odd integer. Show that for every positive integer n the number $a^{2^n} - 1$ has at least $n + 1$ distinct prime divisors.
8	The plane has been partitioned into N regions by three bunches of parallel lines. What is the least number of lines needed in order that $N > 1981$?
9	For a function $f : [0, 1] \to [0, 1]$ we define $f^1 = f$ and $f^{n+1}(x) = f(f^n(x))$ for $0 \leq x \leq 1$ and $n \in N$. Given that there is a n such that $\lvert f^n(x) - f^n(y)\rvert < \lvert x - y\rvert$ for all distinct $x, y \in [0, 1]$,

prove that there is a unique $x_0 \in [0,1]$ such that $f(x_0) = x_0$.