**Austrian-Polish Competition 1982**

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| – | Individual |
| 1 | Find all pairs $(n,m)$ of positive integers such that $gcd((n+1)^m - n, (n+1)^{m+3} - n) > 1$. |
| 2 | Let $F$ be a closed convex region inside a circle $C$ with center $O$ and radius $1$. Furthermore, assume that from each point of $C$ one can draw two rays tangent to $F$ which form an angle of $60^o$. Prove that $F$ is the disc centered at $O$ with radius $1/2$. |
| 3 | If $n \geq 2$ is an integer, prove the equality $$\prod_{k=1}^{n} \tan \frac{\pi}{3}\left(1+\frac{3^k}{3^n-1}\right) = \prod_{k=1}^{n} \cot \frac{\pi}{3}\left(1-\frac{3^k}{3^n-1}\right)$$ |
| 4 | Let $P(x)$ denote the product of all (decimal) digits of a natural number $x$. For any positive integer $x_1$, define the sequence $(x_n)$ recursively by $x_{n+1} = x_n + P(x_n)$. Prove or disprove that the sequence $(x_n)$ is necessarily bounded. |
| 5 | Show that [0,1] cannot be partitioned into two disjoints sets A and B such that B=A+a for some real a. |
| 6 | An integer $a$ is given. Find all real-valued functions $f(x)$ defined on integers $x \geq a$, satisfying the equation $f(x+y) = f(x)f(y)$ for all $x, y \geq a$ with $x + y \geq a$. |
| – | Team |
| 7 | Find the triple of positive integers $(x,y,z)$ with $z$ least possible for which there are positive integers $a, b, c, d$ with the following properties:<br>(i) $x^y = a^b = c^d$ and $x > a > c$<br>(ii) $z = ab = cd$<br>(iii) $x + y = a + b$. |
| 8 | Let $P$ be a point inside a regular tetrahedron ABCD with edge length $1$. Show that $$d(P,AB)+d(P,AC)+d(P,AD)+d(P,BC)+d(P,BD)+d(P,CD) \geq \frac{3}{2}\sqrt{2}$$ , with equality only when $P$ is the centroid of $ABCD$.<br>Here $d(P,XY)$ denotes the distance from point $P$ to line $XY$. |

**9** Define $S_n = \sum_{j,k=1}^{n} \frac{1}{\sqrt{j^2+k^2}}$.
Find a positive constant $C$ such that the inequality $n \leq S_n \leq Cn$ holds for all $n \geq 3$.
(Note. The smaller $C$, the better the solution.)