Austrian-Polish Competition 1983

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–	Individual
1	Nonnegative real numbers a, b, x, y satisfy $a^5+b^5 \leq 1$ and $x^5+y^5 \leq 1$. Prove that $a^2x^3+b^2y^3 \leq 1$.
2	Find all triples of positive integers (p, q, n) with p and q prime, such that $p(p+1)+q(q+1) = n(n+1)$.
3	A bounded planar region of area S is covered by a finite family F of closed discs. Prove that F contains a subfamily consisting of pairwise disjoint discs, of joint area not less than $S/9$.
4	The set N has been partitioned into two sets A and B. Show that for every $n \in N$ there exist distinct integers $a, b > n$ such that $a, b, a+b$ either all belong to A or all belong to B.
5	Let $a_1 < a_2 < a_3 < a_4$ be given positive numbers. Find all real values of parameter c for which the system $x_1+x_2+x_3+x_4 = 1$ $a_1x_1+a_2x_2+a_3x_3+a_4x_4 = c$ $a_1^2x_1+a_2^2x_2+a_3^2x_3+a_4^2x_4 = c^2$ has a solution in nonnegative (x_1, x_2, x_3, x_4) real numbers.
6	Six straight lines are given in space. Among any three of them, two are perpendicular. Show that the given lines can be labeled $\ell_1, ..., \ell_6$ in such a way that ℓ_1, ℓ_2, ℓ_3 are pairwise perpendicular, and so are ℓ_4, ℓ_5, ℓ_6.
–	Team
7	Let P_1, P_2, P_3, P_4 be four distinct points in the plane. Suppose $\ell_1, \ell_2, , \ell_6$ are closed segments in that plane with the following property: Every straight line passing through at least one of the points P_i meets the union $\ell_1 \cup \ell_2 \cup \cup \ell_6$ in exactly two points. Prove or disprove that the segments ℓ_i necessarily form a hexagon.
8	(a) Prove that $(2^{n+1}-1)!$ is divisible by $\prod_{i=0}^{n}(2^{n+1-i}-1)^{2^i}$, for every natural number n (b) Define the sequence (c_n) by $c_1 = 1$ and $c_n = \frac{4n-6}{n}c_{n-1}$ for $n \geq 2$. Show that each c_n is an integer.
9	To each side of the regular p-gon of side length 1 there is attached a $1 \times k$ rectangle, partitioned into k unit cells, where k and p are given positive integers and p an odd prime. Let P be the resulting nonconvex star-like polygonal figure consisting of $kp+1$ regions (kp unit cells and the p-gon). Each region is to be colored in one of three colors, adjacent regions having different

colors. Furthermore, it is required that the colored figure should not have a symmetry axis. In how many ways can this be done?

