Austrian-Polish Competition 1987

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– Individual

1 Three pairwise orthogonal chords of a sphere S are drawn through a given point P inside S. Prove that the sum of the squares of their lengths does not depend on their directions.

2 Let n be the square of an integer whose each prime divisor has an even number of decimal digits. Consider $P(x) = x^n - 1987x$. Show that if x, y are rational numbers with $P(x) = P(y)$, then $x = y$.

3 A function $f : R \to R$ satisfies $f(x + 1) = f(x) + 1$ for all x. Given $a \in R$, define the sequence (x_n) recursively by $x_0 = a$ and $x_{n+1} = f(x_n)$ for $n \geq 0$. Suppose that, for some positive integer m, the difference $x_m - x_0 = k$ is an integer. Prove that the limit $\lim_{n\to\infty} \frac{x_n}{n}$ exists and determine its value.

4 Does the set $\{1, 2, 3, ..., 3000\}$ contain a subset A consisting of 2000 numbers that $x \in A$ implies $2x \notin A$?!! :?:

5 The Euclidian three-dimensional space has been partitioned into three nonempty sets A_1, A_2, A_3. Show that one of these sets contains, for each $d > 0$, a pair of points at mutual distance d.

6 Let C be a unit circle and $n \geq 1$ be a fixed integer. For any set A of n points $P_1, ..., P_n$ on C define $D(A) = \underset{d}{max}\, \underset{i}{min} \delta(P_i, d)$, where d goes over all diameters of C and $\delta(P, \ell)$ denotes the distance from point P to line ℓ. Let F_n be the family of all such sets A. Determine $D_n = \underset{A \in F_n}{min} D(A)$ and describe all sets A with $D(A) = D_n$.

– Team

7 For any natural number $n = \overline{a_k...a_1a_0}$ $(a_k \neq 0)$ in decimal system write $p(n) = a_0 \cdot a_1 \cdot ... \cdot a_k$, $s(n) = a_0 + a_1 + ... + a_k$, $n^* = \overline{a_0a_1...a_k}$. Consider $P = \{n | n = n^*, \frac{1}{3}p(n) = s(n) - 1\}$ and let Q be the set of numbers in P with all digits greater than 1.
(a) Show that P is infinite.
(b) Show that Q is finite.
(c) Write down all the elements of Q.

8 A circle of perimeter 1 has been dissected into four equal arcs B_1, B_2, B_3, B_4. A closed smooth non-selfintersecting curve C has been composed of translates of these arcs (each B_j possibly occurring several times). Prove that the length of C is an integer.

9 Let M be the set of all points (x, y) in the cartesian plane, with integer coordinates satisfying $1 \leq x \leq 12$ and $1 \leq y \leq 13$.
(a) Prove that every 49-element subset of M contains four vertices of a rectangle with sides parallel to the coordinate axes.
(b) Give an example of a 48-element subset of M without this property.

