Austrian-Polish Competition 1988

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by parmenides51, fallinlovewithmaths

–	Individual
1	Let $P(x)$ be a polynomial with integer coefficients. Show that if $Q(x) = P(x) + 12$ has at least six distinct integer roots, then $P(x)$ has no integer roots.
2	If $a_1 \le a_2 \le .. \le a_n$ are natural numbers ($n \ge 2$), show that the inequality $$\sum_{i=1}^{n} a_i x_i^2 + 2\sum_{i=1}^{n-1} x_i x_{i+1} > 0$$ holds for all n-tuples $(x_1, ..., x_n) \ne (0, ..., 0)$ of real numbers if and only if $a_2 \ge 2$.
3	In a ABCD cyclic quadrilateral 4 points K, L ,M, N are taken on AB , BC , CD and DA , respectively such that KLMN is a parallelogram. Lines AD, BC and KM have a common point. And also lines AB, DC and NL have a common point. Prove that KLMN is rhombus.
4	Determine all strictly increasing functions $f : R \to R$ satisfying $f(f(x) + y) = f(x + y) + f(0)$ for all $x, y \in R$.
5	Two sequences $(a_k)_{k\ge 0}$ and $(b_k)_{k\ge 0}$ of integers are given by $b_k = a_k + 9$ and $a_{k+1} = 8b_k + 8$ for $k \ge 0$. Suppose that the number 1988 occurs in one of these sequences. Show that the sequence (a_k) does not contain any nonzero perfect square.
6	Three rays h_1, h_2, h_3 emanating from a point O are given, not all in the same plane. Show that if for any three points A_1, A_2, A_3 on h_1, h_2, h_3 respectively, distinct from O, the triangle $A_1A_2A_3$ is acute-angled, then the rays h_1, h_2, h_3 are pairwise orthogonal.
–	Team
7	Each side of a regular octagon is colored blue or yellow. In each step, the sides are simultaneously recolored as follows: if the two neighbors of a side have different colors, the side will be recolored blue, otherwise it will be recolored yellow. Show that after a finite number of moves all sides will be colored yellow. What is the least value of the number N of moves that always lead to all sides being yellow?
8	We are given 1988 unit cubes. Using some or all of these cubes, we form three quadratic boards A, B, C of dimensions $a \times a \times 1$, $b \times b \times 1$, and $c \times c \times 1$ respectively, where $a \le b \le c$. Now we place board B on board C so that each cube of B is precisely above a cube of C and B does

not overlap C. Similarly, we place A on B. This gives us a three-floor tower. What choice of a, b and c gives the maximum number of such three-floor towers?

9 For a rectangle R with integral side lengths, denote by $D(a, b)$ the number of ways of covering R by congruent rectangles with integral side lengths formed by a family of cuts parallel to one side of R. Determine the perimeter P of the rectangle R for which $\frac{D(a,b)}{a+b}$ is maximal.