Austrian-Polish Competition 1989

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–	Individual
1	Show that $(\sum_{i=1}^n x_iy_iz_i)^2 \le (\sum_{i=1}^n x_i^3)(\sum_{i=1}^n y_i^3)(\sum_{i=1}^n z_i^3)$ for any positive reals x_i, y_i, z_i.
2	Each point of the plane is colored by one of the two colors. Show that there exists an equilateral triangle with monochromatic vertices.
3	Find all natural numbers N (in decimal system) with the following properties: (i) $N = \overline{aabb}$, where $\overline{aab}$ and $\overline{abb}$ are primes, (ii) $N = P_1P_2P_3$, where $P_k (k = 1, 2, 3)$ is a prime consisting of k (decimal) digits.
4	Let P be a convex polygon in the plane. Show that there exists a circle containing the entire polygon P and having at least three adjacent vertices of P on its boundary.
5	Let A be a vertex of a cube ω circumscribed about a sphere k of radius 1. We consider lines g through A containing at least one point of k. Let P be the intersection point of g and k closer to A, and Q be the second intersection point of g and ω. Determine the maximum value of $AP \cdot AQ$ and characterize the lines g yielding the maximum.
6	A sequence $(a_n)_{n\in N}$ of squares of nonzero integers is such that for each n the difference $a_{n+1} - a_n$ is a prime or the square of a prime. Show that all such sequences are finite and determine the longest sequence.
–	Team
7	Functions $f_0, f_1, f_2, ...$ are recursively defined by $f_0(x) = x$ and $f_{2k+1}(x) = 3^{f_{2k}(x)}$ and $f_{2k+2} = 2^{f_{2k+1}(x)}$, $k = 0, 1, 2, ...$ for all $x \in R$. Find the greater one of the numbers $f_{10}(1)$ and $f_9(2)$.
8	ABC is an acute-angled triangle and P a point inside or on the boundary. The feet of the perpendiculars from P to BC, CA, AB are A', B', C' respectively. Show that if ABC is equilateral, then $\frac{AC'+BA'+CB'}{PA'+PB'+PC'}$ is the same for all positions of P, but that for any other triangle it is not.
9	Find the smallest odd natural number N such that N^2 is the sum of an odd number (greater than 1) of squares of adjacent positive integers.