Austrian-Polish Competition 1991

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by parmenides51

–	Individual
1	Show that there are infinitely many integers $m \ge 2$ such that $\binom{m}{2} = 3\binom{n}{4}$ holds for some integer $n \ge 4$. Give the general form of all such m.
2	Find all solutions (x, y, z) to the system $(x^2-6x+13)y = 20$ $(y^2-6y+13)z = 20$ $(z^2-6z+13)x = 20$.
3	Given two distinct points A_1, A_2 in the plane, determine all possible positions of a point A_3 with the following property: There exists an array of (not necessarily distinct) points $P_1, P_2, ..., P_n$ for some $n \ge 3$ such that the segments $P_1P_2, P_2P_3, ..., P_nP_1$ have equal lengths and their midpoints are $A_1, A_2, A_3, A_1, A_2, A_3, ...$ in this order.
4	Let $P(x)$ be a real polynomial with $P(x) \ge 0$ for $0 \le x \le 1$. Show that there exist polynomials $P_i(x) (i = 0, 1, 2)$ with $P_i(x) \ge 0$ for all real x such that $P(x) = P_0(x) + xP_1(x)(1-x)P_2(x)$.
5	If x, y, z are arbitrary positive numbers with $xyz = 1$, prove the inequality $$x^2 + y^2 + z^2 + xy + yz + zx \ge 2(\sqrt{x} + \sqrt{y} + \sqrt{z})$$.
6	Suppose that there is a point P inside a convex quadrilateral $ABCD$ such that the triangles PAB, PBC, PCD, PDA have equal areas. Prove that one of the diagonals bisects the area of $ABCD$.
–	Team
7	For a given positive integer n determine the maximum value of the function $f(x) = \frac{x+x^2+...+x^{2n-1}}{(1+x^n)^2}$ over all $x \ge 0$ and find all positive x for which the maximum is attained.
8	Consider the system of congruences $xy \equiv -1 \pmod{z}$, $yz \equiv 1 \pmod{x}$, $zx \equiv 1 \pmod{y}$. Find the number of triples (x, y, z) of distinct positive integers satisfying this system such that one of the numbers x, y, z equals 19.
9	For a positive integer n denote $A = \{1, 2, ..., n\}$. Suppose that $g : A \to A$ is a fixed function with $g(k) \neq k$ and $g(g(k)) = k$ for $k \in A$. How many functions $f : A \to A$ are there such that $f(k) \neq g(k)$ and $f(f(f(k))) = g(k)$ for $k \in A$?