Austrian-Polish Competition 1993

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–	Individual
1	Solve in positive integers x, y the equation $2^x - 3^y = 7$.
2	Consider all tetrahedra $ABCD$ in which the sum of the areas of the faces ABD, ACD, BCD does not exceed 1. Among such tetrahedra, find those with the maximum volume.
3	Define $f(n) = n + 1$ if $n = p^k > 1$ is a power of a prime number, and $f(n) = p_1^{k_1} + ... + p_r^{k_r}$ for natural numbers $n = p_1^{k_1}...p_r^{k_r}$ ($r > 1, k_i > 0$). Given $m > 1$, we construct the sequence $a_0 = m, a_{j+1} = f(a_j)$ for $j \geq 0$ and denote by $g(m)$ the smallest term in this sequence. For each $m > 1$, determine $g(m)$.
4	The Fibonacci numbers are defined by $F_0 = 1, F_1 = 1, F_{n+2} = F_{n+1} + F_n$. The positive integers A, B are such that A^{19} divides B^{93} and B^{19} divides A^{93}. Show that if $h < k$ are consecutive Fibonacci numbers then $(AB)^h$ divides $(A^4 + B^8)^k$
5	Solve in real numbers the system $$\begin{cases} x^3 + y = 3x + 4 \\ 2y^3 + z = 6y + 6 \\ 3z^3 + x = 9z + 8 \end{cases}$$
6	If $a, b \geq 0$ are real numbers, prove the inequality $$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leq \frac{a + \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + b}{4} \leq \frac{a + \sqrt{ab} + b}{3} \leq \sqrt{\left(\frac{a^{2/3} + b^{2/3}}{2}\right)^3}$$ For each of the inequalities, find the cases of equality.
–	Team
7	The sequence (a_n) is defined by $a_0 = 0$ and $a_{n+1} = [\sqrt[3]{a_n + n}]^3$ for $n \geq 0$. (a) Find a_n in terms of n. (b) Find all n for which $a_n = n$.

8 Determine all real polynomials $P(z)$ for which there exists a unique real polynomial $Q(x)$ satisfying the conditions $Q(0) = 0$, $x + Q(y + P(x)) = y + Q(x + P(y))$ for all $x, y \in R$.

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9 Point P is taken on the extension of side AB of an equilateral triangle ABC so that A is between B and P. Denote by a the side length of triangle ABC, by r_1 the inradius of triangle PAC, and by r_2 the exradius of triangle PBC opposite P. Find the sum $r_1 + r_2$ as a function in a.
