Austrian-Polish Competition 1995

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– Individual

1 Determine all real solutions $(a_1, ..., a_n)$ of the following system of equations:

$$\begin{cases} a_3 = a_2 + a_1 \\ a_4 = a_3 + a_2 \\ ... \\ a_n = a_{n-1} + a_{n-2} \\ a_1 = a_n + a_{n-1} \\ a_2 = a_1 + a_n \end{cases}$$

2 Let $X = \{A_1, A_2, A_3, A_4\}$ be a set of four distinct points in the plane. Show that there exists a subset Y of X with the property that there is no (closed) disk K such that $K \cap X = Y$.

3 Let $P(x) = x^4 + x^3 + x^2 + x + 1$. Show that there exist two non-constant polynomials $Q(y)$ and $R(y)$ with integer coefficients such that for all $Q(y) \cdot R(y) = P(5y^2)$ for all y .

4 Determine all polynomials $P(x)$ with real coefficients such that $P(x)^2 + P\left(\frac{1}{x}\right)^2 = P(x^2)P\left(\frac{1}{x^2}\right)$ for all x.

5 ABC is an equilateral triangle. A_1, B_1, C_1 are the midpoints of BC, CA, AB respectively. p is an arbitrary line through A_1. q and r are lines parallel to p through B_1 and C_1 respectively. p meets the line B_1C_1 at A_2. Similarly, q meets C_1A_1 at B_2, and r meets A_1B_1 at C_2. Show that the lines AA_2, BB_2, CC_2 meet at some point X, and that X lies on the circumcircle of ABC.

6 The Alpine Club organizes four mountain trips for its n members. Let E_1, E_2, E_3, E_4 be the teams participating in these trips. In how many ways can these teams be formed so as to satisfy $E_1 \cap E_2 \neq \varnothing$, $E_2 \cap E_3 \neq \varnothing$, $E_3 \cap E_4 \neq \varnothing$?

– Team

7 Consider the equation $3y^4 + 4cy^3 + 2xy + 48 = 0$, where c is an integer parameter. Determine all values of c for which the number of integral solutions (x, y) satisfying the conditions (i) and (ii) is maximal:
(i) $|x|$ is a square of an integer;
(ii) y is a squarefree number.

8 Consider the cube with the vertices at the points $(\pm 1, \pm 1, \pm 1)$. Let $V_1, ..., V_{95}$ be arbitrary points within this cube. Denote $v_i = \overrightarrow{OV_i}$, where $O = (0, 0, 0)$ is the origin. Consider the 2^{95} vectors of the form $s_1 v_1 + s_2 v_2 + ... + s_{95} v_{95}$, where $s_i = \pm 1$.
(a) If $d = 48$, prove that among these vectors there is a vector $w = (a, b, c)$ such that $a^2 + b^2 + c^2 \leq 48$.
(b) Find a smaller d (the smaller, the better) with the same property.

9 Prove that for all positive integers n, m and all real numbers $x, y > 0$ the following inequality holds:

$$(n-1)(m-1)(x^{n+m} + y^{n+m}) + (n+m-1)(x^n y^m + x^m y^n) \geq nm(x^{n+m-1} y + x y^{n+m-1}).$$